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ABSTRACTS

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Contents

Graph of the polytope for COMPLETE BIPARTITE SUBGRAPH problem

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Consider the well-known NP-complete problem COMPLETE BIPARTITE SUBGRAPH (CBS) [1].

Given a complete edge-weighted graph G with vertex set $\mathbb{N}_n = \{1, \ldots, n\}$ and a positive integer k, $1 \leqslant k \leqslant n$. Required to find a complete bipartite subgraph of graph G with the total number of vertices is equal to k, such that the total weight of its edges would have been maximal.

Note that when $k = n$ the problem becomes a classical problem of maximum cut (see [1, 2]).

We define polytope of CBS problem. Consider the space $\mathbb{R}^m$, where m equals the number of edges in G, ie $m = n(n-1)/2$. For each complete bipartite subgraph η of G, the total number of vertices is equal to k, we consider its characteristic vector $x = x(\eta)$. Let X be a set of all such x. Polytope M of our problem is convex hull of X: $M = \operatorname{conv} X$. Clearly, X is the set of vertices of polytope M. The set of edges is described by the following

Theorem 1. *Vertices x_1, x_2 ($x_i = x(\eta_i), i = \{1, 2\}$) of polytope M are non-adjacent if and only if corresponding bipartite subgraph η_1 and η_2 have equal part, while other parts differ by at least two vertices.*

To prove Theorem 1, we can use the following statement: x_1 and x_2 are non-adjacent if and only if there exists $x \in X, x \neq x_1, x \neq x_2$ such that the inequality $x \leqslant x_1 + x_2$ is satisfied.

Using Theorem 1 one can obtain an exponential lower bound for the clique number of graph G of polytope M.

Theorem 2. *The clique number of graph G of polytope M is such, that inequality is satisfied*

$$p(X) \geqslant [n/k] 2^{\sqrt{k}/2-2}.$$

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Convex Pentagons Which Tile the Plane

Olga Bagina*

Abstract

We consider the problem of classifying the convex pentagons that edge-to-edge tile the plane. We demonstrate that in any of such tiling there exists a pentagon with one of the following degree lists of its vertices: (3,3,3,3,3),(3,3,3,3,4),(3,3,3,3,5),(3,3,3,3,6),(3,3,3,4,4). This enables us to perform a sorting that provides a complete description of such pentagons.

1 Introduction

Suppose there exists an infinite family of regularly closed planar sets (i.e. the closures of their interiors in the standard topology of the plane) of the same shape. If there is a cover of the plane by such sets without overlapping interiors, we say they tile the plane. Such a cover of the plane is called a *monohedral tiling* and the basic tile is called its *prototile*. A polygon P *tiles the plane*, if there exists a tiling with polygons congruent to P.

There are many symmetric tilings of the plane, that is, those invariant under some motions of the plane, that form their symmetry groups. If the symmetry group of a tiling has two non-collinear translations, then the tiling is called *periodic*, and *aperiodic* otherwise. If the symmetry group of a tiling acts transitively on the set of its tiles, the tiling is said to be *isohedral* or *transitive* on its tiles. Quite often the edge-to-edge tilings are considered as well. For every pair of tiles of such a tiling either
(i) the tiles are disjoint, or
(ii) the tiles have exactly one common vertex, or else
(iii) the tiles have exactly one common edge.

All the types of isohedral edge-to-edge tilings have been described in [3] and [4]. If the symmetry group of a tiling acts transitively on a block of $k > 1$ tiles, the tiling is said to be k-block transitive or k-isohedral. The problem of finding and classifying all polygonal tiles with arbitrary tilings is still open. Such tilings include both k-block transitive ($k \geqslant 1$) and aperiodic tilings. Any

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triangle or quadrangle can tile the plane. On the other hand, it is known that the convex n-gons with $n > 6$ can not tile the plane. Tilings of the plane by hexagons have been studied and described completely by K. Reinhardt in his doctoral thesis [6]. They split into three categories.

The problem of finding a complete description of convex pentagons that tile the plane remains open in spite of numerous attempts. Fourteen types of such pentagons have been discovered but it is still unknown whether the list is complete.

2 Main Results

Let us denote the consecutive vertices and angles of a convex pentagon by X_0, X_1, X_2, X_3, X_4 and x_0, x_1, x_2, x_3, x_4 respectively, and let $C_i = |X_{i-1}X_i|$, be its side lengths, $i = 0, 1, 2, 3, 4 \mod 5$. The following types of convex pentagons that tile the plane have been found so far:
1. $x_0 + x_1 = 180°$;
2. $x_0 + x_2 = 180°, C_1 = C_3$;
3. $x_0 = x_2 = x_3 = 120°, C_0 = C_1, C_3 = C_2 + C_4$;
4. $x_0 = x_2 = 90°, C_0 = C_1, C_2 = C_3$;
5. $x_2 = 2x_0 = 120°, C_0 = C_1, C_2 = C_3$;
6. $x_1 + x_3 = 180°, x_0 = 2x_3, C_0 = C_1 = C_2, C_3 = C_4$;
7. $x_0 + 2x_3 = 360°, x_2 + 2x_1 = 360°, C_0 = C_1 = C_2 = C_3$;
8. $x_1 + 2x_0 = 360°, x_2 + 2x_3 = 360°, C_0 = C_1 = C_2 = C_3$;
9. $x_1 + 2x_4 = 360°, x_2 + 2x_3 = 360°, C_0 = C_1 = C_2 = C_3$;
10. $x_4 = 90°, x_0 + x_3 = 180°, 2x_1 - x_3 = 180°, 2x_2 + x_3 = 360°, C_0 = C_4 = C_1 + C_3$;
11. $x_0 = 90°, x_2 + x_4 = 180°, 2x_1 + x_2 = 360°, C_3 = C_4 = 2C_0 + C_2$;
12. $x_0 = 90°, x_2 + x_4 = 180°, 2x_1 + x_2 = 360°, 2C_0 = C_3 = C_2 + C_4$;
13. $x_0 = x_2 = 90°, 2x_1 = 2x_4 = 360° - x_3, C_2 = C_3, 2C_2 = C_4$;
14. $x_3 = 90°, x_0 + x_2 = 180°, x_0 + 2x_4 = 360°, C_0 = 2C_2 = 2C_4$.

The complete description of convex pentagons edge-to-edge tiling the plane have been provided in [1].

Theorem 1 (O. Bagina, [1]). *Convex pentagons edge-to-edge tiling the plane are exactly of the following types:*
1. $x_0 + x_1 = 180°, C_0 = C_2$ *or* $C_3 = C_4$;
2. $x_0 + x_2 = 180°, C_1 = C_3, C_0 = C_2$;
3. $x_0 = x_2 = 90°, C_0 = C_1, C_2 = C_3$;
4. $x_2 = 2x_0 = 120°, C_0 = C_1, C_2 = C_3$;

5. $x_1 + x_3 = 180°, x_0 = 2x_3, C_0 = C_1 = C_2, C_3 = C_4$;
6. $x_0 + 2x_3 = 360°, x_2 + 2x_1 = 360°; C_0 = C_1 = C_2 = C_3$;
7. $x_1 + 2x_0 = 360°, x_2 + 2x_3 = 360°, C_0 = C_1 = C_2 = C_3$;
8. $x_1 + 2x_4 = 360°, x_2 + 2x_3 = 360°, C_0 = C_1 = C_2 = C_3$.

A similar classification have been found independently in [7].

The proof of Theorem 1 includes a complete sorting based on the following result. The *degree* of a vertex of a pentagon P in a tiling is the number of pentagons adjacent to it. Let $(\alpha_0, \ldots, \alpha_4)$ be the degree list of the vertices of P in increasing order.

Theorem 2 (O. Bagina, [1]). *In every edge-to-edge tiling of the plane with convex pentagons there is at least one pentagon with one of the following degree lists of its vertices:*
$(3, 3, 3, 3, 3)$, $(3, 3, 3, 3, 4)$, $(3, 3, 3, 3, 5)$, $(3, 3, 3, 3, 6)$, $(3, 3, 3, 4, 4)$.

A pentagonal prototile P of a tiling with one of the above five degree lists is said to be *central*.

Denote by T_k the set of pentagons whose edges and angles satisfy relations in part k, $k = 1, \ldots, 8$, of Theorem 1.

We introduce the concept of the type $\delta(P)$ of a pentagon P, that can easily be understood, if we illustrate it by an example: $\delta(P) = 11212$ means that $C_0 = C_1 = C_3, C_2 = C_4, C_0 \neq C_2$. There are exactly 12 distinct values of $\delta(P)$: 12345, 11234, 11232, 12134, 12123, 11213, 11212, 11223, 11123, 11122, 11112, 11111.

The type 11111 has been considered in [2]. In 1985 Hunt and Hirschhorn gave the first proof of completeness of the known list of equilateral convex pentagons tiling the plane. An alternative shorter demonstration of this result has been provided in my paper [2].

A corona for a tile P is a minimal set of tiles congruent to P and satisfying the following conditions:
(i) the tiles of the set tile some region V of the plane;
(ii) P is contained in the interior of V.

For the existence of a pentagonal tiling it is necessary, but not sufficient, the existence of a corona for each pentagon of the tiling. The corona search for a central tile is performed successively for each type $\delta(P)$.

The first 9 types have been considered in [1]. For these types the results are stated in the following theorem

Theorem 3 (O. Bagina, [1]). *If P is a pentagon of type $\delta(P) = 12345$, then it does not have any corona. If the type of a pentagonal tile P of a tiling is one of*

the following: 11234, 11232, 12134, 12123, 11213, 11212, 11223, 11123, *then either* P *is in one of the sets* T_k, $k = 1, 2, 3, 4$ *or it has one of the following angle lists:*
1. $x_0 = 180° - x_4, x_1 = 90° + x_4/2, x_2 = 180° - x_4, x_3 = 90° + x_4/2$;
2. $x_0 = 90°, x_1 = 135°, x_2 = x_4 = 112,5°, x_3 = 90°$;
3. $x_0 = x_2 = 120°, x_1 = 150°, x_3 = 45°, x_4 = 105°$;
4. $x_0 = x_2 = 120°, x_1 = 160°, x_3 = 40°, x_4 = 100°$.

The treatment of the remaining two types 11122 and 11112 will be published elsewhere. The results obtained are stated in Theorems 4 and 5 below.

Theorem 4. *Let* P *be a pentagon of a tiling with* $\delta(P) = 11122$, *then either* $P \in T_1 \cup T_3 \cup T_4 \cup T_5$ *or* P *has one of the following angle lists:*
1. $x_0 = x_1 = x_3 = 120°$, $x_2 = x_4 = 90°$;
2. $x_0 = 140°$, $x_1 = 80°$, $x_2 \approx 117,88°$, $x_3 = 120°$, $x_4 \approx 82,12°$;
3. $x_0 = 120°$, $x_1 \approx 84,74°$, $x_2 = 120°$, $x_3 = 120°$, $x_4 \approx 95,26°$;
4. $x_0 \approx 141,33°$, $x_1 \approx 77,34°$, $x_2 \approx 102,66°$, $x_3 \approx 154,67°$, $x_4 \approx 64°$;
5. $x_0 \approx 141,33°$, $x_1 \approx 77,34°$, $x_2 \approx 122°$, $x_3 \approx 116°$, $x_4 \approx 83,33°$;
6. $x_0 = 150°$, $x_1 = 90°$, $x_2 = 105°$, $x_3 = 120°$, $x_4 = 75°$.

Theorem 5. *Let* P *be a pentagon of a tiling with* $\delta(P) = 11112$. *Then either* $P \in T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_6 \cup T_7 \cup T_8$ *or it has one of the following 14 angle lists:*
1. $x_0 \approx 93,1°, x_1 = 120°, x_2 \approx 133,5°, x_3 \approx 73,5°, x_4 \approx 120°$;
2. $x_0 \approx 94,6°, x_1 \approx 113,6°, x_2 \approx 132,7°, x_3 \approx 75,9°, x_4 \approx 123,2°$;
3. $x_0 \approx 88,71°, x_1 \approx 135,65°, x_2 \approx 74,78°, x_3 \approx 135,65°, x_4 \approx 105,22°$;
4. $x_0 \approx 81,2°, x_1 \approx 139,4°, x_2 \approx 69,7°, x_3 \approx 139,4°, x_4 \approx 110,3°$;
5. $x_0 = x_1 = x_3 = 120°, x_2 \approx 94,34°, x_4 \approx 85,66°$;
6. $x_0 \approx 108,28°, x_1 \approx 125,86°, x_2 \approx 78,05°, x_3 \approx 140,98°, x_4 \approx 86,83°$;
7. $x_0 \approx 129,13°, x_1 = 90°, x_2 \approx 101,74°, x_3 = 135°, x_4 \approx 84,13°$;
8. $x_0 \approx 126,42°, x_1 \approx 77,86°, x_2 \approx 107,15°, x_3 \approx 141,07°, x_4 \approx 87,49°$;
9. $x_0 \approx 83,3°, x_1 = 120°, x_2 \approx 78,3°, x_3 \approx 138,4°, x_4 = 120°$;
10. $x_0 \approx 124,23°, x_1 \approx 82,82°, x_2 \approx 111,54°, x_3 \approx 124,23°, x_4 \approx 97,18°$;
11. $x_0 \approx 75,96°, x_1 \approx 142,02°, x_2 \approx 66,05°, x_3 \approx 142,02°, x_4 \approx 113,95°$;
12. $x_0 \approx 85,88°, x_1 \approx 137,06°, x_2 \approx 68,53°, x_3 \approx 145,74°, x_4 \approx 102,79°$;
13. $x_0 \approx 128,22°, x_1 = x_4 \approx 85,48°, x_2 \approx 103,56°, x_3 \approx 137,26°$;
14. $x_0 = 360° - 2x_3, x_1 = x_3, x_2 = 180° - x_4$.

Theorem 6. *In all of the 24 special cases mentioned in Theorems 3-5 above the pentagon* P *has a corona, but the corona can not be extended to a tiling.*

Furthermore, one can derive from Theorems 3-5 the following

Corollary 7. *A convex pentagon P satisfying the conditions*
(i) P has a corona,
(ii) $P \notin T_k, k = 1, \ldots, 8$, and
(iii) P has one of the following degree lists of its vertices: $(3,3,3,3,3)$, $(3,3,3,3,4)$, $(3,3,3,3,5)$, $(3,3,3,3,6)$, $(3,3,3,4,4)$,
has one of the 24 angle lists given in Theorems 3-5.

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Morse theory and numerical algorithms for computation of topological characteristics of three-dimensional bodies

Yaroslav Bazaikin* Iskander Taimanov†

Abstract

We present an algorithm of computation of the topological characteristics of three-dimensional bodies using a discrete version of the Morse theory.

In the modern oil industry oil reservoirs are modelled as the excursion sets of the permeability function and a problem arises to compare different realizations of the same oil reservoir. Topology of excursion set (which is a three-dimensional cubed body) can be a criterion of such comparison.

To estimate the topological structure of three-dimensional cubed body $M \subset \mathbb{R}^3$ we look at a change of level sets of the diagonal function $f(x_1, x_2, x_3) = x_1 + x_2 + x_3$ and investigate those vertices of M which change the topology of level set. In this approach apart from non-degenerate critical points one more degenerate point appears: "the monkey saddle".

In addition we consider a discrete version of gradient flow and define the boundary operator in the chains group generated over $\mathbb{Z}_2$ by critical points.

The algorithm based on this construction was used for a computation of the Betti numbers of real oil reservoirs models.

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Unavoidable crossings in a thinnest covering of the plane and of the sphere

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Abstract

Two convex disks K and L in the plane are said to *cross each other* if the removal of their intersection causes each disk to fall into disjoint components. Almost all major theorems from the early seventies concerning the covering density of a convex disk were proved only for crossing-free coverings. We will show that there is no hope to prove the general theorem by finding suitable rossing free rearrangement. Several coverings containing unovoidable crossings will be presented here, both in the plane and on the sphere.

The central problem of the theory of packing and covering is to find the most economical ways to pack (*i.e.*, to arrange without overlap) congruent copies of a given disk within a given convex domain or to cover the domain by congruent copies of a given disk. Results of the packing and covering theory are surveyed in Brass, Moser and Pach [1]. One of the earliest conjectures concerning covering of the entire plane is still open:

Conjecture 1 (L. Fejes Tóth [3]). *If d is the density of a covering of the plane with congruent copies of a convex disk C, then $d \geqslant \frac{\mathrm{Area}(C)}{\mathrm{Area}(h)}$, where h denotes a hexagon of maximum area inscribed in C.*

The above inequality was proved only for a special case, namely under the restriction that the congruent copies *do not cross each other* Removing the troubling "no crossings allowed" restriction is one of the major unsolved problems in this area.

We start by a negative result, concerning coverings of bounded regions. Heppes covered a square with two crossing hexagons (see Figure 1) and with a fairly elaborate case analysis showed that there is only one way to cover the square with them. Later by a series of examples of Wegner [2] presented coverings of *bounded* convex domains, with a large number of hexagons, where the desired rearrangements are impossible.

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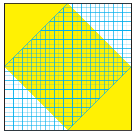
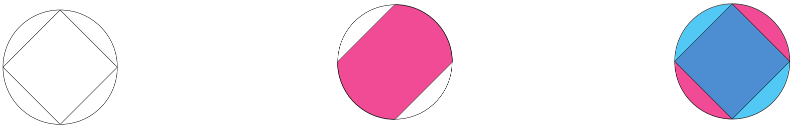


Figure 1: Covering the unit square with two hexagons.

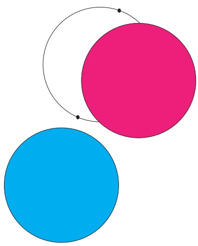
Remark 1. Somewhat surprisingly one can modify Heppes's idea so that uniqueness of covering is almost trivial:



Unit circle with an inscribed square. | Curved quadrilateral, one of the two pieces. | Two crossing curved quadrilaterals

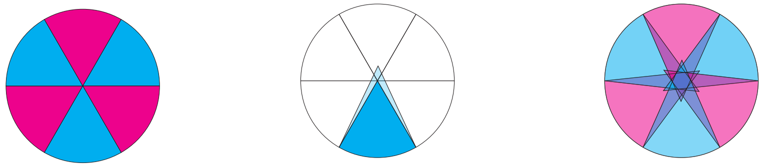
Figure 2: Covering the unit circle with two curved quadrilaterals.

Proof. The following is a 'proof without words' argument to show that even if we replace the curved quadrilaterals with full circles, in order to cover the unit disc one of them has to be concentric with it:



Remark 2. One can construct a covering of a unit circle with arbitrary large number of congruent pieces and prove with a simple argument that crossing free rearrangements do not exist.

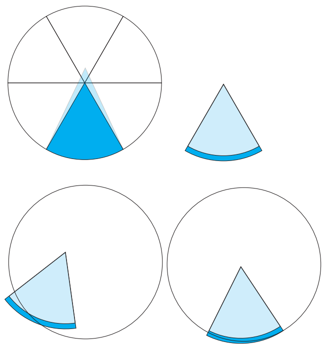
Proof. The following is almost a 'proof without words' argument to show uniqueness of the covering. All what one has to notice is that each piece has



Unit circle partitioned into 6 sectors. Stretching each sector to include the center. Some pairs of the stretched sectors cross each other.

Figure 3: Covering the unit circle with $n = 6$ stretched sectors.

to be at its original position in order to cover $\frac{1}{n}$-th of the full perimeter of the unit circle:



Next we describe a covering of the plane with unavoidable crossings. Three perpendiculars drawn from the center of a regular hexagon to its three non-adjacent sides partition the hexagon into three congruent pentagons. Obviously, the plane can be tiled by such pentagons. But a slight modification produces a (non-tiling) pentagon with an unexpected covering property: every thinnest covering of the plane by congruent copies of the modified pentagon must contain crossing pairs. The example has no bearing on the validity of Fejes Tóth's bound in general, but it shows that any prospective proof must take into consideration the existence of unavoidable crossings.

Dissect a regular hexagon H of area 3 into three congruent pentagons (of unit area, of course) by cutting along segments drawn from the center of H perpendicularly to three nonadjacent sides of H, as shown in Figure 4a. The resulting pentagonal tile P_0 has two right angles and three angles measuring $120°$ each. Enlarge the pentagon P_0 slightly by moving the vertices at the right angles outside H and so that in the modified pentagon the sides emanating from the relocated vertices change their directions by the same small angle. Denote by ε the increase of the pentagon's area caused by the modification and denote the so enlarged pentagon by P_ε. Obviously, if $\varepsilon > 0$ is sufficiently small, then P_ε is not a tile. If each of the three pentagons into which H was partitioned is modified in the exactly same way, then the three congruent copies of P_ε cover the regular hexagon H and they *cross each other*, no matter how small value of ε is chosen.

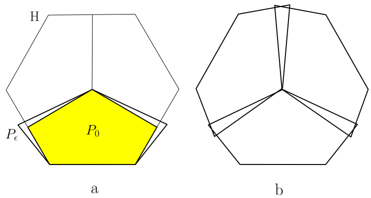


Figure 4: Modified pentagons

Extend hexagon H to a hexagonal tiling of the entire plane. The result is an apparently thin plane covering, of density $1 + \varepsilon$. The following can be proved:

Theorem 1. *If ε is sufficiently small, then every covering of the plane with congruent copies of the pentagon P_ε is of density at least $1+\varepsilon$ and each of the thinnest coverings must contain crossing pentagons.*

Concerning coverings of the sphere we will start with various regular tilings of the sphere. Similarly to most of the planar construction the tiles will be stretched to have crossings.

Theorem 2. *The covering generated by the regular tetrahedron (regular icosahedron resp.) is unique (Figure 5).*

Theorem 3. *The covering generated by the cube is not unique (Figure 6).*

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Figure 5: Partition the faces of the cube and apply stretches at the midpoints of the edges.

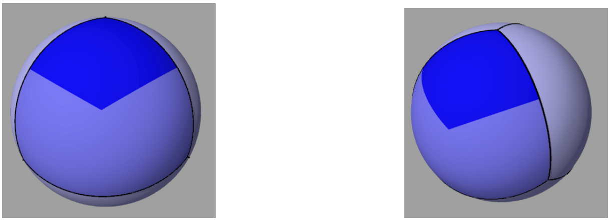

Figure 6: Partition the faces of the tetrahedron and apply stretches at the midpoints of the edges.

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Finding a subset of vectors with a coordinatewise large sum

I.I. Bogdanov* G.R. Chelnokov†

Abstract

We show that for any vectors $\mathbf{u}_1, \dots, \mathbf{u}_N \in \mathbb{R}_+^d$ and any rational $0 \leqslant a = p/q \leqslant 1$ there exists a set of indices $I \subseteq \{1, 2, \dots, N\}$ such that $|I| \leqslant (d-1) + \lceil (pN - d + 1)/q \rceil$, and $\sum_{i \in I} \mathbf{u}_i - a \sum_{i=1}^N \mathbf{u}_i \in \mathbb{R}_+^d$. The bound on $|I|$ is sharp for all $N \geqslant (q-1)(d-1)$.

Definition 1. Let $\mathbf{u}_1, \dots, \mathbf{u}_N \in \mathbb{R}_+^d$ be N vectors with nonnegative coordinates, and let $a \in [0,1]$ be some real number. We say that a set of indices $I \subseteq \{1, \dots, N\}$ is *a-rich* if

$$\sum_{i \in I} \mathbf{u}_i - a \sum_{i=1}^N \mathbf{u}_i \in \mathbb{R}_+^d$$

Definition 2. By $f_{N,d}(a)$ we denote the minimal number f such that for any N vectors $\mathbf{u}_1, \dots, \mathbf{u}_N \in \mathbb{R}_+^d$ there exists an a-rich set I with $|I| \leqslant f$.

Further we consider only rational a and write $a = p/q$ with $q > 0$ and $\gcd(p,q) = 1$.

Using the well-known theorem of Alon on splitting of necklaces [1] one can show that $f_{N,d}(p/q) \leq pN/q + O(pd)$. This bound is asymptotically sharp. A bit better bound $f_{N,d}(p/q) \leq pN/q + O(d)$ can be obtained by using a theorem of Stromquist and Woodall [2] on finding a set where several measures agree. We find an exact value of $f_{N,d}(p/q)$ for almost all values of parameters.

Theorem 1 (Bogdanov and Chelnokov, [3]). *For every positive integer N, d and rational $a = p/q \in [0,1]$, we have*

$$f_{N,d}(p/q) \leq (d-1) + \left\lceil \frac{pN - d + 1}{q} \right\rceil.$$

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Moreover,

$$f_{N,d}(p/q) = (d-1) + \left\lceil \frac{pN-d+1}{q} \right\rceil \quad \textit{for all} \;\; N \geqslant (q-1)(d-1)$$

The proof uses the tools of linear programming.

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On transversals of quasiagebraic families of sets

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Abstract

In this paper we generalize some theorems about transversals of families of finite sets to some cases of families of infinite sets.

In this paper we consider Helly-Gallai numbers for families of sets that are similar to families of sets which are solutions for finite systems of equations.

Definition 1. A set X, $|X| \leq t$, is called a *t-transversal* of a family of sets F if $A \cap X \neq \emptyset$ for every $A \in F$. The Helly-Gallai number $HG(t, F)$ of a family of sets F is called a minimal number k such that if every subfamily $P \subseteq F$ with $|P| \leq k$ has a t-transversal, then the family F has a t-transversal (see [1,2]).

For every family F, the existence of a 1-transversal is equivalent to the condition that the intersection of all sets of F is nonempty. Therefore a number $HG(1, F)$ is called a number of Helly $H(F)$ for a family F.

Remark 3. If F is a family of a convex compact sets in $\mathbb{R}^d, d \geq 2$ and $t \geq 2$, then numbers $HG(t, F)$ don't exist. If F is a family of intervals on the line, then $HG(t, F) = t + 1$.

The numbers of Helly for a family of algebraic varities were found T.S. Motzkin [3].

Definition 2. Let A^d_m be a family of sets of common zeroes in $\mathbb{R}^d$ for a finite collection of polynomials of d variables and degree not grater than m.

Theorem (Motzkin [3])**.**

$$H(A^d_m) = \binom{m+d}{d}.$$

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The Helly–Gallai numbers for algebraic varities A_n^d were determined by M. Deza and P. Frankl [4], and V.Dolnikov [3]. They are given by the formula:

$$HG(t, A_m^d) = \binom{\binom{m+d}{d} + t - 1}{t}.$$

In the papers [5],[6] the Helly–Gallai numbers for families of sets of more general kind were considered. More precisely, they were the zero sets of linear finite dimensional subspaces of functions on a ground set V with coefficients in a field F.

In particular, the Helly–Gallai numbers

$$HG(t, S_{d-1}) = \binom{d+t+1}{t}$$

for families of spheres S_{d-1} in $\mathbb{R}^d$ were found. Independently Helly numbers $H(S_{d-1})$ were found by H. Maehara [7].

Now we give some estimates for the Helly–Gallai numbers of quasialgebraic families of sets.

Definition 3. Let F be a family of sets. Denote inductively

$$F^1 = \{B : B = A_i \cap A_j, \text{ where } A_i, A_j \in F \text{ and } A_i \neq A_j\}$$

and

$$F^{k+1} = \{B : B = A_i \cap A_j, \text{ where } A_i, A_j \in F^k \text{ and } A_i \neq A_j\}.$$

Definition 4. A family of sets F has the $(d, m)-$*property* if $|A| \leq d$ for every $A \in F^m$. If a family F has $(d, m)-$property, then such family is called a quasiagebraic family of a dimension m and a degree d. Denote the class of such families by QA_m^d.

For example, a family of lines in $\mathbb{F}^d$, where $\mathbb{F}$ is a field, or a family of lines of a finite projective plain, or a family of all edges of a graph G or a family for sets of all edges of an arbitrary graph G that contain a given vertex are quasialgebraic families of a dimension 1 and a degree 1. Such families are called linear families in a literature.

The family of circles is a quasiagebraic family of dimension 1 and degree 2. The family of finite sets of cardinality d is a quasialgebraic family of dimension 0 and degree d.

More generally, the family A_m^d is a quasiagebraic family with different a dimension and a degree.

Definition 5. A family of sets F has the $[d,m]-$*property* if $|\cap_{A\in G} A| \le d$ for every $G \subset F, |G| = m$.

Remark 4. The family of hyperplanes in $\mathbb{F}^m$, where $\mathbb{F}$ is an arbitrary field, has $(1,m)$-property, the family of hyperplanes in $\mathbb{F}^m$ in general position has $[1,m]$-property.

The main goal of our paper is to prove the following theorems.

Theorem 1.

$$\binom{d+m+t}{d+m} \le HG(t, QA_m^d) \le \frac{t^{m+d+1}-1}{t-1}.$$

Theorem 2.

$$HG(2, QA_1^d) \le 2d^2 + 3 \textit{ for } d \ge 2.$$

Theorem 3.

$$HG(t, QA_1^1) \le \max\left(t^2 - t + 3, \binom{t+2}{2}\right).$$

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Jammed packings of disks in a torus

Robert Connelly *

Abstract

This is a report, including open questions, on some of the results of a group of students, Alexander Smith, Lisa Piccirillo, John Chiarelli, Jeffrey Shen, and Diwalker Raisingh in a Research Experience for Undergraduates (REU) program at Cornell University in the summer of 2012.

1 Introduction

A union of disks with disjoint interiors is called a *packing*. A *torus* is defined as the quotient $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$ of the Euclidean $\mathbb{R}^2$ plane by the a lattice $\Lambda = \{ng_1 + mg_2 \mid m, n \in \mathbb{Z}\}$, where $g_2, g_2 \in \mathbb{R}^2$. The *packing graph* of a packing in a torus is obtained by joining the centers of touching disks. Such a packing is called *collectively jammed* if the only continuous motion of the disks in $\mathbb{T}^2$ preserving the packing property is a translation. For example, Figure 1 shows a packing of six disks in $\mathbb{T}^2$ which is collectively jammed.

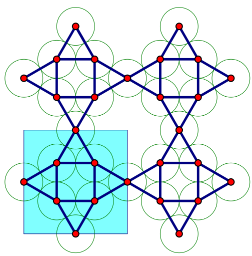

Figure 1: A fundamental region of the torus is shaded, and the graph of the packing of the six disks is indicated. Notice that a seventh disk can be placed rattling around in the octagonal hole.

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2 Problems

There are several problems that are amenable and would benefit from being worked on. See [1] for definitions and basic results.

1. When the generators of the lattice Λ are $g_1 = (1, 0), g_2 = (1/2, \sqrt{3}/2)$, and there are n packing disks in this *triangular torus* $\mathbb{T}^2$, is there a reasonable upper bound for the maximum density δ of the packing, where δ is the ratio of the total area of the disks to the area of $\mathbb{T}^2$.

2. Classify the packing graphs of the collectively jammed packings of the triangular torus $\mathbb{T}^2$, where all the regions are triangles or rhombuses.

3. Find examples of collectively jammed packings with the smallest possible density. Can you find examples with density smaller than $\frac{3}{2}\frac{\pi}{(\sqrt{3}+1)^2}$, the density of the one in Figure 1? Can you find examples of jammed packings that have regions with more than eight edges?

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Efficient Localization of Mesh Nodes in a Delaunay Triangulation

Natalia Dyshkant*

Abstract

We study the problem of a mesh localization in a Delaunay triangulation. The computationally efficient method using minimum spanning tree of Delaunay triangulation constructed on the set of mesh nodes is proposed. We show that the average case complexity of the method is linear by the total number of nodes in the mesh and the triangulation.

The well-known geometric search problem of point location in a Delaunay triangulation [1] is formulated as follows: given a point Q and a Delaunay triangulation T, it is required to declare the triangle of T containing Q. One of the most fast methods to solve this problem (in general case) has $O(\log N)$ computational complexity and $O(N)$ memory usage, where N is the number of nodes in T [2].

A two-dimensional (plane) mesh is a set of mutually connected geometric elements (nodes, edges, and cells). The grid nodes represent a finite set of points on the plane. Let g be a plane mesh of N_1 nodes and T_2 be a Delaunay triangulation constructed on N_2 nodes. In the problem of mesh nodes localization in a Delaunay triangulation it is required to solve point location problem for each node of g. Then the unstructured mass query of N_1 nodes can be processed by time $O(N_1 \log N_2)$.

We show how the Delaunay triangulation constructed on the nodes of g can be used for acquisition of more efficient solution. The proposed method is based on *Euclidean minimum spanning trees* (MST) of Delaunay triangulation which can be constructed in linear time [3]. The method was proposed by the author in [6]. There is approach for point location problem that uses *"walk along a line" strategy* [7]. The idea of the approach consists in gradual transition from some initial point M of known location to source point Q along the straight line (MQ). During each transition step changing on adjacent (neighboring by edge) triangle is implemented. After such stage is finished, a *location path* consisting of adjacent triangulation triangles is constructed (see Fig. 1). Case of belonging of a certain node of T_2 to segment $[MQ]$ is a case of a special interest.

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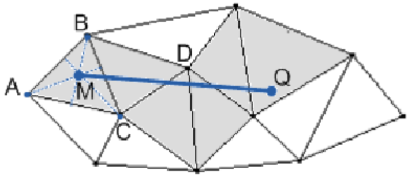

Figure 1: Point location in Delaunay triangulation.

In the proposed method for mesh localization, locations paths pass along the edges of the minimum spanning tree (see Fig. 2). As a spanning tree does not contain cycles and passes through all points of the mesh g, the algorithm will work correctly: it does not loop and performs location of all mesh nodes.

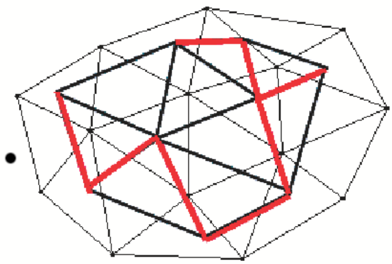

Figure 2: Mesh nodes localization in Delaunay triangulation using MST.

In this work we prove that in case of uniform distribution for nodes of triangulation T_2 and mesh g the average case complexity of the method is linear by $max(N_1, N_2)$. Using some results of Kostuk [4] and Bose and Devroye [5], we show that

Lemma 1. *Suppose g_1, g_2 are plane meshes with the numbers of nodes N_1, N_2, respectively, $N_1/N_2 \leqslant c = const$, and the sets of nodes of the both meshes uniformly distributed uniformly in a rectangular area. Then, the average number of intersections between the MST for nodes of g_1 and edges of the Delaunay triangulation constructed on g_2 is linear by N_2.*

Theorem 2. *Under the conditions of lemma 1, the algorithm for localization of mesh g_1 in the Delaunay triangulation constructed on the set of nodes of g_2, on basis of the MST of g_1 has the average case complexity $O\big(max(N_1, N_2)\big)$.*

The proof of lemma 1 and theorem 2 is omitted.

The assumption of uniform distribution of mesh nodes is appropriate for the majority of practical applications.

In the worst case the mesh localization is not linear. We consider method for construction of simulated example shows that the worst case complexity of the proposed method is quadratic.

In [9, 10] the problem of merging of unseparated Delaunay triangulations were studied: given two Delaunay triangulations T_1 and T_2 constructed on sets g_1 and g_2, respectively, with intersected convex hulls, it is required to construct the *united Delaunay triangulation* T. We say that a triangle of T is an $interface$ triangle if it joins nodes from the both sets g_1, g_2 (see Fig. 3).

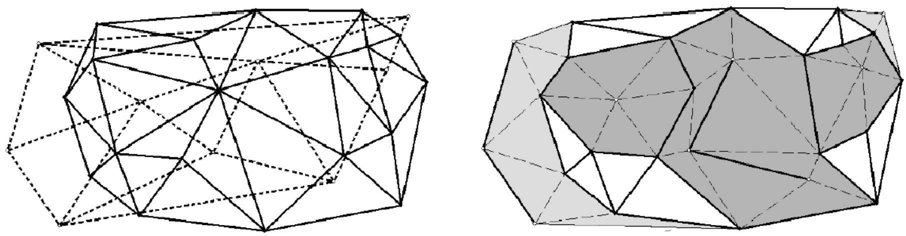

Figure 3: Unseparated triangulated unregular meshes (on the left); united Delaunay triangulation of these grids with darkened interface triangles (on the right).

By N denote the total number of nodes in the sets g_1, g_2. We consider the problem of identifying of all interface triangles of T. Using the results acquired by Mestetskiy and Tsarik in [10], we propose the solution of this problem and show that

Theorem 3. *All interface triangles of* T *can be extracted in linear by* N*.*

The theorem 3 allows to receive the following result:

Theorem 4. *Localization of nodes from* g_1 *in a Delaunay triangulation constructes on* g_2 *on basis of the list of all interface triangles can be performed in* $O(N)$ *in the worst case.*

The proof of the theorems 3, 4 can be found in [11].

The problem of mesh nodes localization is appeared as application for a problem of comparing discrete models of single-valued surfaces given on different unregular meshes. In [8] the author proposed the computationally efficient approach to compute measures for such surfaces which constructs Delaunay triangulations for each of given meshes and locate each node of each mesh in the triangulation constructed on the nodes of the other mesh. The results of the theorem 2 and the theorem 4 were experimentally verified and confirmed by computing experiments on real and model data.

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Methods of fractal and computational geometry for generalization of linear cartographic objects

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Olga Yakimova ‡ Vladislav Alexeev ‡ Victoriya Bogaevskaya ‡

Andrey Gorohov ‡ Margarita Preobrazhenskaya ‡

Abstract

We present a new algorithm for simplifying linear cartographic objects and results obtained with a computer program implementing this algorithm

We consider the problem of generalization of linear cartographic objects [1].

The source data for algorithm is the line string given by vertices – two dimensional points representing some linear geographic object (borderline of the state, river, contour of the continent etc.). It is required to draw the line string to given scale maintaining the most important details of the line.

It is obvious that on relatively even sections of the line we can delete many points without loosing the information while on relatively curved parts we must maintain bigger number of points. The difficulty of the problem consists in formalization of intuitive idea of the important details of the curve.

Our algorithm consists of following steps:

1. **Uniform parametrization.** Transforming the line string to the line with constant edge lengths.

2. **Segmentation.** Separating the line string to segments with similar curvature properties.

3. **Assessment of fractal dimension.** Calculation the fractal dimension of each section.

4. **Simplification.** Simplification of each section with the Douglas-Peucker algorithm [3]. Parameters of simplification are being selected in accordance with fractal dimension of the section.

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5. **Smoothing.** Smoothing the result line string with B-splines.

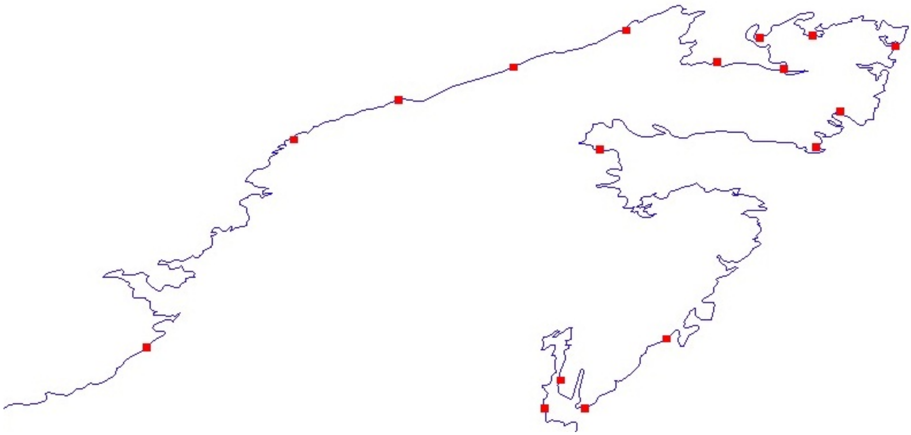

Figure 1: Example of segmentation. Endpoints of segments are marked by square red dots.

The idea of using fractal dimension for cartographic generalization is one of the popular modern approaches [4]. Earlier it was discussed by one of the authors in the book [1].

The main new idea of the algorithm is the procedure of automatic segmentation – finding the parts of the line with similar properties (Step 2 – Segmentation of the algorithm). To decide which sections of the line string must be united in one segment we use procedure including following steps:

1. Dividing the line string into initial segments. The number of the points n in each segment is the parameter of the algorithm. By this time, step 1 (Uniform parametrization) has been already performed, hence, each segment has a form $L = (P_1, \dots, P_n)$ ($P_i \in \mathbf{R}^2$), where $|P_{i+1} - P_i| = \rho$, $\forall i = 1, \dots, n-1$.

2. For each point P_i, $(1 < i < n)$ of the segment L we approximate the curvature C_i, defined as a complement of the angle α_i (angle at the point P_i) to π, i.e.

$$C_i = \arccos \frac{(x_{i+1} - x_i)(x_i - x_{i-1}) + (y_{i+1} - y_i)(y_i - y_{i-1})}{\rho^2}.$$

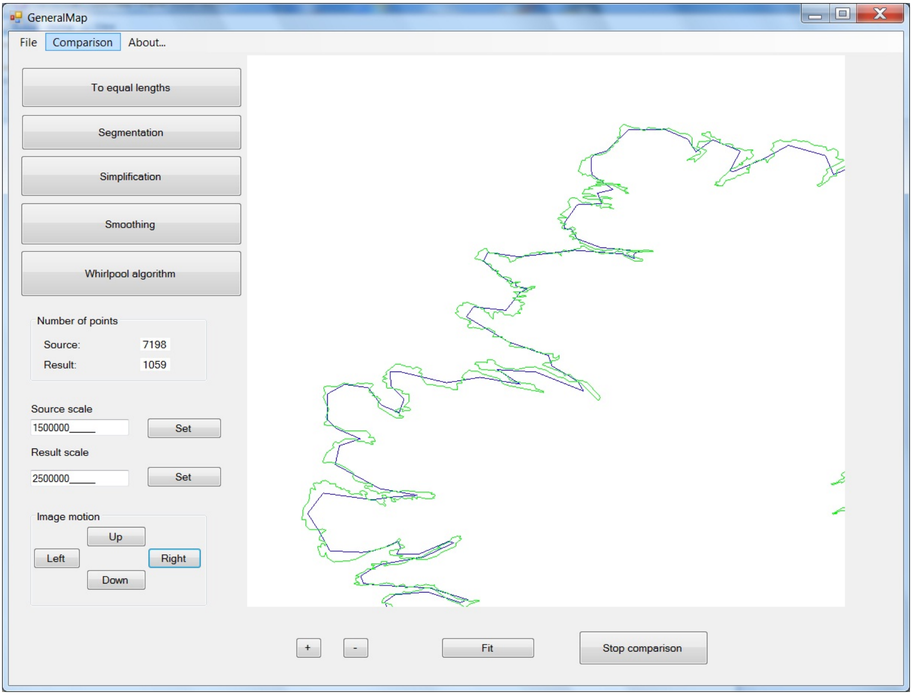

Figure 2: Result of generalization with our program. Green line – original line string, blue line – result of generalization.

3. Calculating the total curvature of the current segment $C(L) = \sum_{1<i<n} |C_i|$ and the winding number $W(L) = C(L)/(2\pi)$.

4. Finding the number of curvature extremes $E(L)$ – the number of extremas of the function $C : L \to \mathbf{R}$, $C(P_i) = C_i$ $(i = 1, \ldots, n)$, i.e. number of vertices P_i such that $C_i > C_{i-1}$, $C_i > C_{i+1}$ or $C_i < C_{i-1}$, $C_i < C_{i+1}$.

5. Calculating the integral characteristic of the segment $M = 2W(L) + fE(L)$, where f is a real number (parameter of the algorithm).

6. We unite the segments with minimal differences of the characteristic M while the number of segments is greater than some given number N_s and number of points on each segment is less than some given number N_{max}.

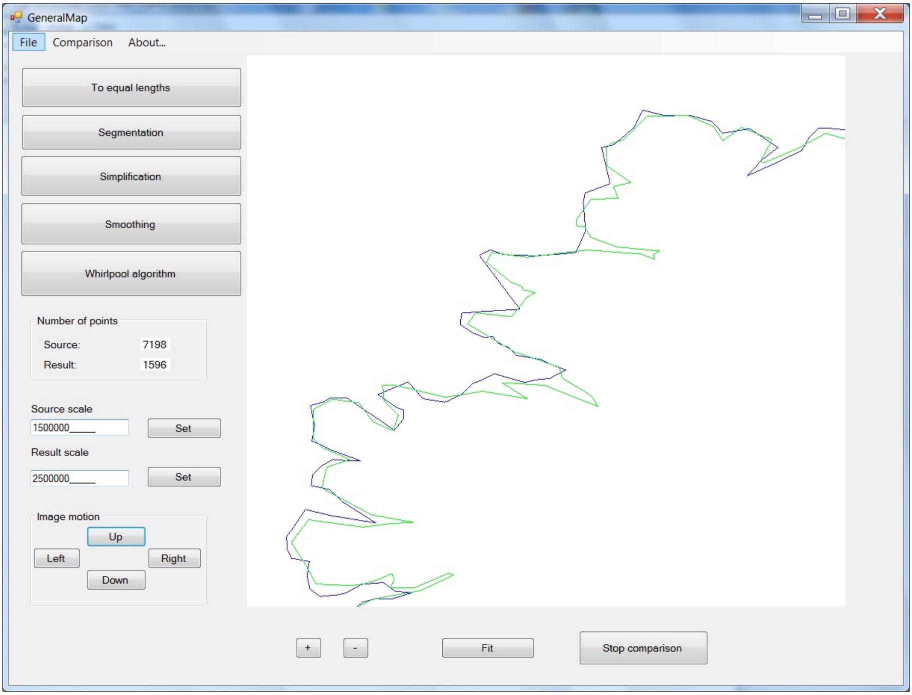

Figure 3: Comparison of our result (green line) with result of the Whirlpool algorithm (blue line).

The Figure 1 shows the example of segmentation produced by this algorithm.

The computer program implementing this algorithm was developed. The parameters which must be set manually are original scale and resulting scale. Other parameters of the algorithm (n – initial number of points on a segment, N_{max} – maximal number of points on a segment, N_s – minimal number of segments, f – weight of for the number of extremas) can be set by default values or chosen by user.

The Figure 2 shows the result of work of our algorithm for the line string of 7198 points. The sale was changed from 1:1500000 to 1:2500000. We used default values of parameters: $n = 5$, $N_{max} = 198$, $N_s = 133$, $f = 0.5$. In this case our algorithm decreased the number of points from 7198 to 1059.

The comparison of results with other known algorithms shows some advantages of the proposed method. The Figure 3 shows the comparison of our algorithm (green line) with the Whirlpool algorithm [2] (blue line). The Whirlpool algorithm was applied to the same line string with the same scale transformation as was used for the Figure 2. The Whirlpool algorithm maintained 1596 points of 7198. Our algorithm represent the shape of the borderline more accurately while maintaining lesser number of points.

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Sabitov polynomials for volumes of four-dimensional polyhedra

Alexander A. Gaifullin*

Abstract

We generalize the Sabitov theorem on computation of the volume of the polyhedron in the 3-space from its combinatorial type and edge lengths to the 4-dimensional case.

It is well-known that a triangle in the Euclidean plane is rigid. This means that it remains congruent to itself whenever we deform it so that the edge lengths remain constant. Besides, the area of a triangle can be computed from its edge lengths by the classical Heron formula. On the other hand any polygon with at least 4 edges is not rigid. Hence, its area cannot be computed from its edge lengths.

It is an interesting problem how these simple facts can be generalized to higher dimensions. The famous Cauchy theorem claims that each convex simplicial polytope is rigid. In the end of the 19th century R. Bricard constructed his famous flexible octahedra. These are (non-embedded) octahedra in $\mathbb{R}^3$ that can be deformed continuously so that the edge lengths are constant, but the polyhedron does not remain congruent to itself in the process of deformation. An example of an embedded flexible polyhedron in $\mathbb{R}^3$ was constructed by R. Connelly only in 1977. Surprisingly, it appeared that, for all known examples, the volume of the flexible polyhedron remains constant. It was posed as a conjecture, called *Bellows Conjecture*, that this phenomenon actually holds for every flexible polyhedron in $\mathbb{R}^3$. Bellows Conjecture was proved in 1996 by I. Kh. Sabitov [3], [4], see also [1]. His approach was to show that the volume of the polyhedron of given combinatorial type with given edge lengths can take only finitely many values. More precisely, his result was as follows.

Theorem 1 (I. Kh. Sabitov, [3]). *The volume of an arbitrary simplicial polyhedron in the 3-dimensional Euclidean space is a root of a monic polynomial whose coefficients depends on the combinatorial type and the edge lengths of the polyhedron only:*

$$V^N + a_1(\ell)V^{N-1} + \ldots + a_N(\ell) = 0,$$

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where by ℓ we denote the set of lengths of edges of the polyhedron. Besides, the coefficients $a_j(\ell)$ are polynomials in the squares of lengths of edges of the polyhedron.

Since then it has been unknown whether the same holds in arbitrary dimension $n \geq 3$. We prove the direct analog of Theorem 1 for polyhedra in the 4-dimensional Euclidean space. Thus our result is as follows.

Theorem 2 (A. A. Gaifullin, [2]). *The volume of an arbitrary simplicial polyhedron in the 4-dimensional Euclidean space is a root of a monic polynomial whose coefficients depends on the combinatorial type and the edge lengths of the polyhedron only:*

$$V^N + a_1(\ell)V^{N-1} + \ldots + a_N(\ell) = 0,$$

where by ℓ we denote the set of lengths of edges of the polyhedron. Besides, the coefficients $a_j(\ell)$ are polynomials in the squares of lengths of edges of the polyhedron.

Corollary 3 (A. A. Gaifullin, [2]). *The volume of each flexible polyhedron in the 4-dimensional Euclidean space remains constant under the flex.*

Our proof contains two principally new aspects with respect to the 3-dimensional case. The first one is the extension of the notion of a polyhedron based on the concept of a simplicial cycle. The second one has algebraic geometrical nature. We use certain new lemma concerning the properties of the variety consisting of possible sets of edge lengths for the polyhedra of the given combinatorial type.

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Finite-dimensional models of diffusion chaos

Sergey Glyzin*

Abstract

Some parabolic systems of the reaction-diffusion type exhibit the phenomenon of diffusion chaos. Specifically, when the diffusivities decrease proportionally, while the other parameters of a system remain fixed, the system exhibits a chaotic attractor whose dimension increases indefinitely.

Reaction–diffusion systems are parabolic boundary value problems of the form

$$\frac{\partial u}{\partial t} = \nu D \Delta u + F(u), \quad \left.\frac{\partial u}{\partial \vec{n}}\right|_{\partial\Omega} = 0, \tag{1}$$

where Δ is the Laplacian; $u \in \mathbb{R}^k$, $k \geq 2$; $D = \operatorname{diag}\{d_1, \ldots, d_k\}$, $d_j > 0$, $j = 1, \ldots, k$; $\nu > 0$ is a parameter responsible for a proportional decrease in the diffusivities; $\vec{n}$ is the outward normal to the sufficiently smooth boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^m$, $m \geq 1$; and $F(u)$ is a smooth vector function. It is well known that these systems serve as mathematical models of many biophysical and ecological processes (see, e.g., [1]). A typical situation is that the point model corresponding to system (1), i.e., the system of ordinary differential equations

$$\dot{u} = F(u), \tag{2}$$

has an exponentially orbitally stable cycle $u = u_0(t)$, $du_0/dt \neq 0$ of period $T_0 > 0$. An example of system (1) is a mathematical model of Belousov's reaction (see [2])

$$\dot{\mathscr{N}}_1 = r_1[1+a(1-\mathscr{N}_3)-\mathscr{N}_1]\mathscr{N}_1, \ \dot{\mathscr{N}}_2 = r_2[\mathscr{N}_1-\mathscr{N}_2]\mathscr{N}_2, \ \dot{\mathscr{N}}_3 = r_3[\mathscr{N}_2-\mathscr{N}_3]\mathscr{N}_3 \tag{3}$$

in the most important range of its parameters $r_j > 0$, $j = 1, 2, 3$ and $a > 0$.

Another example is the well-known Hutchinson equation (see [3])

$$\dot{\mathscr{N}} = r[1 - \mathscr{N}(t-1)]\mathscr{N} \tag{4}$$

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with $r > \pi/2$, which describes the oscillations in the mammal population density. The presence or absence of a delay is not important for the issues discussed below.

It is easy to see that cycle $u_0(t)$ is also exhibited by distributed model (1) (i.e., it solves the corresponding boundary value problem). To examine its stability properties, the equation in (1) is linearized about this cycle and the Fourier method in terms of the eigenfunctions of the Laplacian is applied to the resulting linear boundary value problem. As a result, we obtain the system

$$\dot{h} = [A_0(t) - zD]h, \tag{5}$$

where $A_0(t) = F'(u)|_{u=u_0(t)}$; the parameter z takes the discrete values $\nu\lambda_k$, $k = 0, 1, \ldots$; and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ are the eigenvalues of the operator $-\Delta$ with Neumann boundary conditions arranged in increasing order.

We assume that z in (5) varies continuously on the half-line $z \geq 0$. Let $\mu_s = \mu_s(z)$, $s = 1, \ldots, k$ denote the multipliers of system (5), and define

$$\alpha(z) = \max_{1 \leq s \leq k} \left\{ \frac{1}{T_0} \mathrm{Re} \ln \mu_s(z) \right\}. \tag{6}$$

It is always true that $\alpha(0) = 0$, since system (5) at $z = 0$ (assuming that cycle $u_0(t)$ in point model (2) is exponentially stable) has a simple unit multiplier associated with the Floquet solution $h = \dot{u}_0(t)$, while all the other multipliers lie in the disk $\{\mu \in \mathbb{C} : |\mu| < 1\}$. Below, we need the following definition.

Definition 1. The parabolic boundary value problem (1) is said to be biological or belong to the class **B** if the following constraints hold:

(1a) The corresponding point model (2) has an exponentially orbitally stable cycle $u_0(t)$.

(1b) There are $0 \leq z_1 < z_2$ such that function (6) is strictly positive on the interval $z_1 < z < z_2$.

(1c) For all sufficiently small $\nu > 0$, the dynamical system generated by problem (1) in the phase space $C(\overline{\Omega}; \mathbb{R}^k)$, $\overline{\Omega} = \Omega \cup \partial\Omega$, has a chaotic attractor A_ν whose Lyapunov dimension $d_L(A_\nu)$ tends to $+\infty$ as $\nu \to 0$.

As was noted above, condition (1a) is typical of boundary value problems (1) arising in various biophysical and ecological applications (which explains the coinage of the term 'biological system'). The condition $\alpha(z) > 0$ for $z \in (z_1, z_2)$ in (1b) ensures that cycle $u_0(t)$ of distributed model (1) is unstable for all sufficiently small ν. Condition (1c) is the most important of the three and guarantees that the phenomenon of diffusion chaos occurs as $\nu \to 0$. In this context, it should be noted that the term 'chaotic attractor' has various interpretations. To be definite, we use the concept of chaos accepted in [4]. The final remark is that, in principle, the Lyapunov dimension in (1c) can be replaced by the Hausdorff or any other one. For the convenience of the subsequent numerical analysis, we use the Lyapunov dimension, assuming that $d_L(A_\nu)$ is defined in terms of the characteristic numbers of the attractor A_ν by the Kaplan–Yorke well-known formula. Based on Definition 1, the concept of diffusion chaos is easily formulated as follows.

Conjecture (on diffusion chaos). The class **B** of parabolic systems (1) is not empty.

Obviously, this conjecture resembles Kolmogorov's well-known hypotheses that the dimension of the attractors of the Navier–Stokes equations increases with the Reynolds number. Moreover, obvious parallelism can be observed between condition (1c) and the Landau–Sell scenario of turbulence development (its mathematical aspects are presented in [5]).

Various finite-dimensional models of diffusion chaos are considered that represent chains of coupled ordinary differential equations and similar chains of discrete mappings. A numerical analysis suggests that these chains with suitably chosen parameters exhibit chaotic attractors of arbitrarily high dimensions.

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A Numerical Approach to the Curve-Skeletons Problem

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Abstract

We propose a novel approach to skeletonization of 3D objects. The skeletonization process is reduced to the numerical optimization problem. The method provides a strict way to evaluate and compare various skeletons of the same 3D object. We describe a particular implementation of our approach and discuss the results of experiments.

A skeleton of a shape is a graph that captures major topological and metrical properties of the shape. It may be considered as an 1D thinning of the original shape. Skeletons are very useful in the computer vision since it's much easier to extract shape's features from a graph rather from its boundary description. Skeleton of a two-dimensional shape is usually defined as the shape's medial axis. Medial axis is the set of all points having more than one closest point on the shape's boundary. 2D medial axis is always an 1D set and can be computed efficiently. However, 3D medial axis contains 2D sheets and therefore is not a graph.

An 1D skeleton of a 3D shape (which is also called a curve-skeleton) is known as an ill-defined object [2]. There is no common strict definition recognized by significant number of papers on the curve-skeletons. Instead of this, curve-skeleton is usually defined as an object produced by the particular algorithm. Different algorithms produce skeleton graphs of different nature and with different properties. Examples of various types of skeletonization algorithms can be found in [2] [3]. This makes it impossible to compare different algorithms with each other. The only possible method of such comparison is an evaluation of visual quality and computational performance.

In this research we develop an approach that allows to evaluate various skeletons. We give a very broad definition of a curve-skeleton as any continuous thinning of the original object. Then we describe the procedure of reconstruction that allows to produce 3D shape from an 1D skeleton (we call it a silhouette). The measure of similarity between the original shape and skeleton's silhouette is an evaluation of the skeleton's quality. Thus the problem

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of skeletonization can be formulated as a numerical optimization problem: to construct a skeleton of the object means to find the best approximation for the object by the skeleton's silhouette. We describe our implementation of this idea which includes the measure of similarity, algorithm for finding the first approximation and numerical methods for the minimization process.

As mentioned above, there's a great variety of definitions of a 3D curve-skeleton. The common idea of these definitions is that the curve-skeleton is a thinned 1D representation of 3D object [2]. It can be formalized in terms of homotopy.

Let Ω be an embedded in $\mathbb{R}^3$ connected open set with boundary $\partial\Omega$ and $\overline{\Omega}$ is its closure:

$$\overline{\Omega} = \Omega \cup \partial\Omega. \tag{7}$$

Let $\Gamma \subset \overline{\Omega}$ be a 3D representation of some graph G such that every edge of G is mapped into some smooth curve in $\mathbb{R}^3$.

Definition 1. Γ is a curve-skeleton of Ω if there's a continuous function

$$H : [0;1] \times \partial\Omega \to \overline{\Omega} \tag{8}$$

such that

$$H(0,x) = x, H(1,\partial\Omega) = \Gamma. \tag{9}$$

The function

$$\sigma(x) \equiv H(1,x) \tag{10}$$

is called a skeleton mapping.

The skeleton mapping describes the correspondence between the surface of the object and its skeleton.

One of the advantages of the 2D skeleton is the ability to recover the original shape from the skeleton. That is possible due to the distance transform function which defines "width" of the shape for every skeleton point: the shape is a union of discs with centers situated on the skeleton branches and radii defined by the distance transform. We need some analogue of the 2D distance transform in order to preserve the reconstruction possibility for 3D curve-skeletons.

Definition 2. A radial function r is a non-negative real-valued function defined on a curve-skeleton:

$$r : \Gamma \to \mathbb{R}, r(x) \geq 0 \ \forall x \in \Gamma. \tag{11}$$

Definition 3. A silhouette of a curve-skeleton Γ with a radial function r is a set

$$S(\Gamma, r) = \bigcup_{x \in \Gamma} B_{r(x)}(x) \tag{12}$$

where $B_r(x)$ is a ball in $\mathbb{R}^3$:

$$B_r(x) = \{y \in \mathbb{R}^3 : \rho(x, y) \leq r(x)\}. \tag{13}$$

The single curve with its silhouette is called a fat curve [1]. Fat curves can be used for the approximation of tubular objects [4].

The silhouette can be considered as a reconstruction of the original 3D object. Unlike 2D skeleton, 3D silhouette is merely an approximation of Ω. So we need a numerical measure of similarity between Ω and $S(\Gamma, r)$. In this work we propose the following function. Let Γ be a curve-skeleton of Ω with a radial function r. An approximation error of (Γ, r) is a value

$$\mathcal{E}(\Omega, \Gamma, r) = \int_{x \in \partial\Omega} \Big(\rho^2\big(x, \sigma(x)\big) - r^2\big(\sigma(x)\big)\Big)^2 dS. \tag{14}$$

The main idea of our method is a numerical optimization of the curve-skeleton in order to minimize the approximation error (14). This approach suggests that there's a first approximation of the curve-skeleton. It can be constructed by almost any existing skeletonization algorithm.

We construct the initial curve-skeleton using Reeb graphs. Reeb graphs are widely used in topological shape analysis (see examples [5][6]). This approach is rather effective and produces skeletons which are always topologically correct (which is important since it's difficult to change topology of the skeleton during the numerical optimization process). Other important advantage of Reeb graphs is that the skeleton mapping σ is generated explicitly. We define a continuous scalar function

$$f : \partial\Omega \to \mathbb{R} \tag{15}$$

called a mapping function. Let L be a topological space of contour lines of f. In other words, L is a quotient space $\partial\Omega / \sim_f$, where $\sim_f$ is an equivalence relation such that

$$x \sim_f y \Leftrightarrow \exists l \subset \partial\Omega : x, y \in l, f(z) \equiv Const\ \forall z \in l, \tag{16}$$

and l is a connected curve in $\partial\Omega$. L is a graph-like space holding the topology of Ω. It just needs to be embedded into $\mathbb{R}^3$ in order to generate a 3D representation of a graph.

Our method is implemented for 3D objects represented by polygonal meshes. Let M be a triangulated closed surface with a set of vertices V and a set of edges E. We assume that V is uniform and detailed enough. The integral in (14) is approximated with a discrete sum $\tilde{\mathcal{E}}$ over all vertices from V.

To avoid confusion with the mesh and its vertices and edges, in this section we'll use the term "joint" for a curve-skeleton vertex and the term "bone" for an edge. Let J be the set of all joints and B be the set of all bones of the curve-skeleton Γ. In our implementation, all bones are straight line segments and all radial functions are linear. When the first approximation is generated, the problem is reduced to a minimization of a function $\tilde{\mathcal{E}}$. It's a polynomial of degree 4 with $4|J|$ variables (since we have $|J|$ joints and each is defined by 4 variables).

We use gradient descent to minimize $\tilde{\mathcal{E}}$. Let J_k be a $4|J|$-dimensional vector describing the set of joints computed on the step k. Then J_{k+1} is equal to

$$J_{k+1} = J_k - \lambda \nabla \tilde{\mathcal{E}}(J_k), \tag{17}$$

where λ is a solution of the minimization problem

$$\tilde{\mathcal{E}}(J_k - \lambda \nabla \tilde{\mathcal{E}}(J_k)) \to \min_{\lambda \in \mathbb{R}}. \tag{18}$$

It's a polynomial of degree 4 with the only one variable, so it's very easy to find its minimum. The procedure (17) is performed until the condition

$$|\tilde{\mathcal{E}}(J_k) - \tilde{\mathcal{E}}(J_{k+1})| < \varepsilon \tag{19}$$

is satisfied, where $\varepsilon > 0$ is a fixed parameter of the algorithm.

An example of our algorithm's work is shown in Figure 1.

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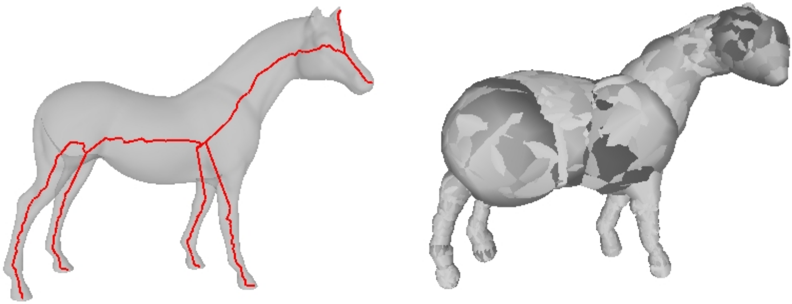

Figure 1: Curve-skeleton of the horse and its silhouette.

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Perfect prismatoids and the conjecture concerning with face numbers of centrally symmetric polytopes

Marina Kozachok*

Abstract

We construct a class of centrally symmetric polytopes – perfect prismatoids and proved some its properties related to the famous conjecture concerning with face numbers of centrally symmetric polytopes. It is proved that any Hanner polytope is a perfect prismatoid and any perfect prismatoid is affinely equivalent to some 0/1-polytope.

A polytope $P \subset \mathbb{R}^d$ is centrally symmetric (or cs, for short) if $P = -P$. We say that faces F and F' of a cs polytope P are *antipodal* if $F = -F'$. Let f_i be a number of i-faces of a cs polytope P. Define $f(P)$ by the total number of non-empty faces of P.

In 1989 Kalai stated the conjecture known as the 3^d-conjecture that every cs d-polytope has at least 3^d non-empty faces:

Conjecture 1. For any cs d-polytope P:

$$f(P) = \sum_{i=0}^{d} f_i(P) \geqslant 3^d.$$

In dimension $d = 3$ the conjecture follows from Euler's relation. In $d = 4$ the conjecture was proved by Ziegler, Werner, Sanyal in 2007. For simplicial and simple d-polytopes the conjecture follows from Stanley's results obtained in 1987.

In $d = 3$ only a cube and an octahedron have 3^3 faces. In $d \geqslant 4$ there exist other polytopes besides a d-cube and crosspolytope with exactly 3^d faces. An important class that attains the bound is the class of Hanner polytopes. It is unknown whether there exist other polytopes with exactly 3^d non-empty faces.

To define Hanner polytopes we introduce the concept of a cross.

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Definition 1. Let $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ be subspaces of $\mathbb{R}^{d_1+d_2}$ and $\mathbb{R}^{d_1} \cap \mathbb{R}^{d_2} = \{O\}$. Let a d_1-dimensional polytope $P_1 \subset \mathbb{R}^{d_1}$ and a d_2-dimensional polytope $P_2 \subset \mathbb{R}^{d_2}$ be centrally symmetric polytopes with the common centre of symmetry O. Then $P = conv\{P_1, P_2\}$. is called a cross of P_1 and P_2. The cross of P_1 and P_2 is denoted by $P_1 \boxtimes P_2$.

Hanner polytopes are defined recursively:

Definition 2. Every centrally symmetric 1-polytope is a Hanner polytope. For dimensions $d \geqslant 2$, a d-polytope is a Hanner polytope if it is the direct product or the cross of two lower dimensional Hanner polytopes.

A centrally symmetric polytope P is called *prismatoid with bases F and F'* if $P = conv(F \cup F')$ where F and F' are antipodal facets of F. Faces of P that do not lie either in F or in F' are called *lateral faces.*

Definition 3. A centrally symmetric convex polytope P is called a *perfect prismatoid* if $P = conv(F \cup F')$ for any pair of its antipodal facets F and F'.

Remark 1. This definition is equivalent to the following one: every pair of antipodal facets contains all vertices of P.

Theorem 1. *Let $P = conv(F \cup F')$ where F and F' are antipodal facets of a cs d-polytope P. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be parallel $(d-2)$-planes supporting F. Let Q_i and Q_i'' be faces of F lying in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Let Q_i and Q_i' be antipodal faces of P. Then for any pair Q_i, Q_i' constructed in such way $conv(Q_i \cup Q_i')$ is a face of P and vice versa: any lateral face of P is $conv(Q_i \cup Q_i')$ for some Q_i and Q_i'.*

In case $d = 3$ the class of perfect prismatoids and the class of Hanner polytopes coincides. (They contain only the cube and octahedron). We prove that in $d > 4$ the class of perfect prismatoids contains the class of Hanner polytopes but does not coincide with it.

Definition 4. A d-polytope is called $0/1$-*polytope* if all its vertices are in $\{0; 1\}^d$.

Theorem 2. *Every d-dimensional perfect prismatoid P is affinely equivalent to some $0/1$-polytope.*

Proof. Let v be some vertex of P and $f_1, \ldots, f_d$ some linearly independent facets containing v. Let $f_1', \ldots, f_d'$ be their antipodal facets. The $2d$ hyperplanes containing these facets form some parallelepiped. Let w be some vertex of P.

For any i, such as $1 \leqslant i \leqslant d$, either f_i or f'_i contains w. So w lies in some d linearly independent facets of the parallelepiped. Hence w is a vertex of the parallelepiped. So P a convex hull of some vertices of the parallelepiped. The parallelepiped is affine equivalent to the unit cube. $\square$

Remind that the direct product of P_1 and P_2 is denoted by $P_1 \times P_2$ and the dual of P by P^*.

Theorem 3. *The dual P^* of a perfect prismatoid P, the direct product $P_1 \times P_2$ and cross $P_1 \boxtimes P_2$ of two perfect prismatoids P_1 and P_2 are perfect prismatoids.*

Proof. First prove that the direct product P of perfect prismatoids P_1 and P_2 is a perfect prismatoid. Any facet F of P is either direct product of some facet F_1 of P_1 and P_2 or vice versa. Assume without loss of generality that $P = F_1 \times P_2$. Let $F'_1 \subset P_1$ and $F' \subset P$ be the antipodal facets to F_1 and F correspondently. Facets F_1 and F'_1 contain all vertices of P_1 since P is a perfect prismatoid. Any vertex of P is a direct product of some vertices of P_1 and P_2, any vertex of F is a direct product of some vertices of F_1 and P_2, any vertex of F' is a direct product of some vertices of F'_1 and P_2. So F and F' contain all vertices of P. Then P is a perfect prismatoid.

Prove that the dual of any perfect prismatoid is a perfect prismatoid. The following properties of a cs polytope are equivavlent:

1. Let F and F' be antipodal facets. Then any vertex lies in either F or F' (this is equivalent to the definition of a perfect prismatoid)

2. Let v and v' be antipodal vertices. Then any facet contains either v or v'.

If some polytope has the first property then its dual has the second one. So the dual of any perfect prismatoid is a perfect prismatoid.

It is known that

$$P_1 \boxtimes P_2 = (P_1^* \times P_2^*)^*$$
$$P_1 \times P_2 = (P_1^* \boxtimes P_2^*)^*.$$

The statement about a cross follows from the first relation and the statements about a direct product and dual proved above. $\square$

The next important theorem follows from Theorem 3.

Theorem 4. *Every Hanner polytope is a perfect prismatoid.*

Proof. The proof is by induction over dimension d. For $d = 1$, there is nothing to prove since any segment is both a perfect prismatoid and a Hanner polytope. Assume the theorem is valid for Hanner d-polytopes and prove it for Hanner $(d+1)$-polytopes. Any $(d+1)$-polytope is either a direct product or a cross of Hanner d-polytopes that are perfect prismatoids by the inductive assumption. So by Theorem 3 any $(d+1)$-polytope is a perfect prismatoid. $\square$

Remark 2. For $d \leqslant 4$ any perfect prismatoid is a Hanner polytope.

But the converse of Theorem 4 is not true.

Theorem 5. *There exist perfect prismatoids that are not Hanner polytopes.*

Proof. Let us construct a perfect 5-prismatoid that is not a Hanner polytope.

Consider two parallel 3-planes $\mathcal{L}_1$ and $\mathcal{L}_2$ containing a tetrahedron $KLMN$ and a triangle ABC correspondently such that $\overrightarrow{AB} = \overrightarrow{KL}$ and $\overrightarrow{BC} = \overrightarrow{MN}$. Let F be a convex hull of these 7 points. Let P be a cs prismatoid over F.

Polytope P contains 14 vertices. We claim that any facet of P is a convex hull of some tetrahedron and triangle such that some opposite edges of the tetrahedron are parallel to some sides of the triangle. So any facet of P contains 7 vertices and thus P is a perfect prismatoid. We can calculate f-vector of P: $(f_0, f_1, f_2, f_3, f_4) = (14, 50, 92, 72, 18)$. So $f(P) = 257 > 3^5$. Thus P is not a Hanner polytope. $\square$

Remark 3. In $d \geqslant 5$ the direct product of P and $d - 5$ segments is a perfect d-prismatoid that is not a Hanner polytope.

Estimating the number of facets of a 2-neighborly polytope

Aleksandr Maksimenko*

Abstract

A d-polytope P is called 2-neighborly if each pair of its vertices forms an edge of P. It is known that combinatorial polytopes are often 2-neighborly (see, e.g., Bondarenko and Maksimenko (2008), Deza and Laurent (1997), Kovalev (2003), Onn (1993)). In addition, Bondarenko and Brodsky (2008) established the following fact. Let P be the convex hull of n randomly choosed vertises of d-dimensional cube, where $n = O(2^{d/6})$. Then, P is 2-neighborly with probability close to 1. Despite these facts, the study of properties of such polytopes so far received little attention. We conjecture that the number $f_0(P)$ of vertices of a 2-neighborly polytope does not exceed the number $f_{d-1}(P)$ of facets. The conjecture is proved for two cases: $d < 7$ and $f_0(P) < d + 6$.

A d-dimensional convex polytope is called a d-polytope. Further i-dimensional faces of a d-polytope are called i-faces. 0-face is a vertex. 1-face is an edge. $(d-1)$-face is called facet. $(d-2)$-face is called ridge. A polytope P is said to be k-neighborly if every set of k or fewer vertices forms a face of P. Let $f_i(P)$ be the number of i-dimensional faces of P.

The problem of estimating $f_i(P)$ in terms of $f_0(P)$ for some classes of polytopes is well known. The upper bound problem was solved in 1970 by McMullen [6]. In particular, the number of facets satisfies the inequality

$$f_{d-1}(P) \le f_{d-1}(C) = \begin{cases} \left(1 + \frac{k}{f_0-k}\right) \cdot \binom{f_0-k}{k}, & \text{for } d = 2k, \\ 2\left(1 - \frac{k}{f_0-k}\right) \cdot \binom{f_0-k}{k}, & \text{for } d = 2k+1, \end{cases}$$

where C is a cyclic polytope with $f_0 = f_0(P)$ vertices. It shows that $f_{d-1}(C) = O(n^{[d/2]})$ for fixed d. Evidently, this bound is tight for 2-neighborly polytopes. The inequality implies the lower bound

$$f_{d-1}(P) \ge \min_C \{f_0(C) : \ f_{d-1}(C) \ge f_0(P)\}.$$

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It is also known that

$$f_{d-1}(P) \geq d+2 \quad \text{for } d+2 \leq f_0(P) \leq 2d.$$

In 1971, Barnette [7] proved the lower bound

$$f_{d-1}(P) \geq (d-1)(f_0(P)-d)+2. \tag{20}$$

for simplicial P. The problem of finding lower bounds for 2-neighborly polytopes has not yet been posed.

Conjecture. *The number $f_0(P)$ of vertices of a 2-neighborly polytope P does not exceed the number $f_{d-1}(P)$ of facets.*

From (20) it follows that the conjecture is true for simplicial 2-neighborly polytopes.

Here we prove the conjecture for two cases: $d \leq 6$ (small dimension) and $f_0(P) \leq d+5$ (small number of vertices).

I. Small dimension

In $\mathbb{R}^3$ the only example of a 2-neighborly polytope is a tetrahedron or a simplex. Some diversity appears only in $\mathbb{R}^4$. Note that every face of a 2-neighborly polytope is a 2-neighborly polytope. Hence, a 2-neighborly 4-polytope is simplicial. So we can use the Dehn-Sommerville equations. For $d=4$ we have

$$f_3 = \frac{f_2}{2} = f_1 - f_0,$$

where f_i – the number of i-faces, $i=0,1,2,3$. Since $f_1 = \dfrac{f_0(f_0-1)}{2}$ for a 2-neighborly polytope, it follows that

$$f_3 = \frac{f_0(f_0-3)}{2} \tag{21}$$

Therefore, the conjecture is true for $d=4$.

For cases $d=5$ and $d=6$ we use the following reasoning. Let F_k be the set of all k-faces of polytope P. For some fixed k we cosider the bipartite graph whose vertices sets is F_{k-1} and F_k. Vertices $x \in F_{k-1}$ and $y \in F_k$ of this graph are adjacent iff $x \subset y$ (here x is $(k-1)$-face and y is k-face). The number $|E|$ of edges of the bipartite graph may be calculated in two ways: go over F_k or go over F_{k-1}. In the first case we have $|E| = \sum\limits_{y \in F_k} f_{k-1}(y)$, where $f_{k-1}(y)$ is the number of $(k-1)$-faces of y. In the second case $|E| \geq (d-k+1)f_{k-1}$, because every $(k-1)$-face of a d-polytope is intersection of at least $(d-k+1)$ of k-faces [8]. Comparing these two results we obtain inequality

$$(d-k+1)f_{k-1} \leq \sum_{y \in F_k} f_{k-1}(y).$$

Since every 3-fase of a 2-neighborly polytope is a simplex, it follows that

$$(d-2)f_2 \leq 4f_3 \tag{22}$$

and

$$(d-1)f_1 \leq 3f_2.$$

For $d=5$ we have

$$4f_1 \leq 3f_2 \leq 4f_3. \tag{23}$$

Now, let's look at the ridge-graph of a polytope. The set of facets of a polytope is the set of vertices of its ridge-graph and the set of ridges of a polytope is the set of edges of its ridge-graph. From (23) it follows that the number of edges of a 2-neighborly 5-polytope does not exceed the number of edges of its ridge-graph. Because the graph of a 2-neighborly polytope is complete, hence the number f_4 of vertices of the ridge-graph greater than or equal the number f_0 of vertices of the graph:

$$f_1 \leq f_{d-2} \Rightarrow f_0 \leq f_{d-1}.$$

Therefore, the conjecture is true for $d=5$.

For $d=6$ we use the Euler's equation [8]

$$f_0 - f_1 + f_2 - f_3 + f_4 - f_5 = 0$$

If we combine this with (22) for $d=6$, we get

$$f_1 - f_0 \leq f_4 - f_5. \tag{24}$$

The graph of a 2-neighborly is complete. Hence $f_1 - f_0 = \dfrac{f_0(f_0-3)}{2}$. On the other hand, the ridge-graph may be uncomplete: $f_4 - f_5 \leq \dfrac{f_5(f_5-3)}{2}$. Using (24), we get

$$f_0(f_0-3) \leq f_5(f_5-3).$$

Therefore, $f_0 \leq f_5$.

II. Small number of vertices

The conjecture is trivial in the case of simplex. We prove

$$f_{d-1} \geq d+5, \quad \text{when } d \geq 4 \text{ and } d+2 \leq f_0 \leq d+5. \tag{25}$$

The proof is by induction on d. For $d = 4$, validity of the inequality follows from (21). Assume that for $d = k$ the proposition is true Let us cosider two cases for the 2-neighborly $(k+1)$-polytope.

Case 1. Assume that every facet of the polytope has exactly $d = k+1$ vertices. Then it is simplicial and we can use (20):

$$f_{d-1}(P) \geq (d-1)(f_0 - d) + 2 \geq 2d \geq d+5 \quad \text{when } f_0 \geq d+2 \text{ and } d \geq 5.$$

Case 2. Assume that the polytope has nonsimplicial facet. So the number s of vertices of the facet satisfies the inequality $k+2 \leq s \leq k+5$. The facet is 2-neighborly. Hence the inequality (25) holds for it and it is adjacent with at least $k+5$ facets of the polytope. This completes the proof of (25).

III. The results of computational experiments

We carried out two different experiments.

In the first one we considered the boolean quadratic polytope [2] for $n = 6$. This is a 21-polytope with 64 vertices. For the convenience of the experiment, there was used its projection P in $\mathbb{R}^{15}$. This projection P is also 2-neighborly and has 57 vertices and 21441 facets. We used the fact that the convex hull of a subset of vertices of a 2-neighborly polytope is also a 2-neighborly polytope. So, we were randomly chosing a subset of vertices of P and countig the number of facets of their convex hull with `lrslib` [9]. There were tested several hundreds of thousands of such polytopes.

In the second experiment we examined all 0/1 polytopes in $\mathbb{R}^6$. For 6 dimension there are $2^{2^6} \approx 18 \cdot 10^{18}$ such polytopes. Among them there were found 322 625 classes of a 2-neighborly polytopes.

The results of processing of these data are summarized in the table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Δ_n	3	4	4	6	10	13	21	26	19	39	65	73	118	144

Where $n = \#\,\mathrm{vert}(Q) - (\dim(Q) + 1)$, $\Delta_n = \min\limits_{Q}\{\#\,\mathrm{facet}(Q) - \#\,\mathrm{vert}(Q)\}$ for all considered 2-neighborly d-polytopes Q.

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Virtual Crossing Numbers for Virtual Knots

Vassily Olegovich Manturov*†

The aim of the talk is to prove that the minimal number of virtual crossings for some families of virtual knots grows quadratically with respect to the minimal number of classical crossings.

The main results of the talk are available at [1].

The main idea of the present paper is to use the *parity arguments*: if there is a smart way to distinguish between *even* and *odd* crossings of a virtual knot so that they behave nicely under Reidemeister moves then there is a way to reduce some problems about *virtual knots* to analogous problems about *their diagrams (representatives).*

Thus, we have to find a certain family of four-valent graph for which the crossing number (minimal number of *additional crossings* (prototypes of virtual crossings) for an immersion in $\mathbb{R}^2$) is quadratic with respect to the number of *vertices* (prototypes of classical crossings).

The study of parity has been first undertaken in [2], see also [3, 4] where functorial mappings from virtual knots to virtual knots were constructed, minimality theorems were proved, and many virtual knot invariants were refined.

In the case of graphs, such families having quadratic growth for the number of additional crossings with respect to the number of the crossings themselves are quite well known to graph theorists: even for trivalent graphs the generic crossing number grows quadratically with respect to the number of vertices, see, e.g., [5].

Then we take an expander family of *odd irreducible framed four-valent graphs* K_n with one unicursal component and with f_n vertices each. Here *odd* means that the corresponding chord diagrams are odd, and irreducible means that no decreasing 2-Reidemeister move can be applied to such a graph. If f_n grows linearly with n, then this family of graphs gives rise to virtual knots with desired properties.

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Skeleton of Binary Image vs Binary Image of Skeleton

Leonid Mestetskiy*

Abstract

Skeleton (or medial axis representation) is a powerful and widely used tool for image shape analysis and classification. In this report we offer the full implementation of the continuous approach to the skeleton construction for binary raster images of any complexity. The approach is based on the approximation of a discrete object by the continuous geometrical figure and the construction of the skeleton for this figure. The aim of the concept of continuous skeleton for raster binary image is to use the correct and elegant model of Voronoi diagram of line segments and polygonal figures to obtain the skeleton of a discrete object. Such application should ensure that, firstly, the correct definition of the skeleton for this object, and secondly, to enable highly efficient use of computational geometry algorithms for obtaining the skeleton.

Skeleton or middle axis of the object is an important descriptor of the image shape. Skeletons are used to feature generation for determine the similarity measures of various shapes in the construction of classifiers. Originally, the concept of skeleton was denoted for continuous objects: the skeleton of a closed region in Euclidean plane is a locus of centers of maximum empty circles in this region. The circle is considered to be empty if all its internal points are internal points of the region. The concept of the skeleton (the middle set of points) was introduced and investigated by Blum [1]. He gave a geometric definition of the skeleton of two-dimensional object, investigated its geometric properties, and demonstrated its usefulness for shape description and shape analysis. However, this concept became the most popular in shape analysis and classification of discrete bitmap digital images. Therefore, the need to generalize the concept of the skeleton for discrete images was raised.

In principle, we can formulate two approaches to extend the concept of the skeleton to discrete images. The first approach, which is the most popular because of ease of implementation, will be called discrete. It consists in a morphological transformation of the original image (Figure 1a) and construction new image (Figure 1b), which can be regarded as a skeleton. In this new

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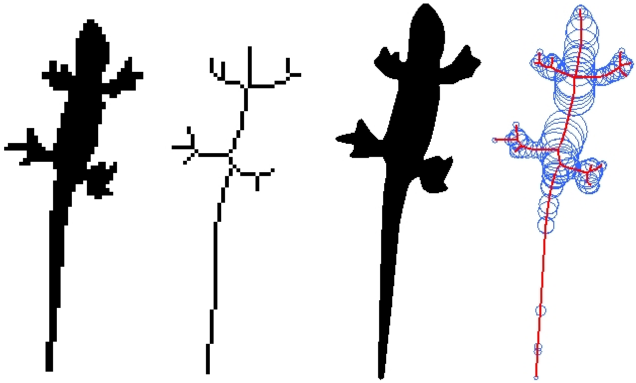

Figure 1: (a) – the discrete binary image, (b) – the discrete skeleton, (c) – the continuous binary image, (d) – the continuous skeleton and inscribed circles.

bitmap medial axis represented by discrete lines one pixel wide. We can say that the resulting image is a digital image of the skeleton.

The discrete approach is implemented in different ways: based on distance maps, thinning, Voronoi diagrams of boundary points [2] [4]. Another approach, which we call continuous, is based on the approximation of a discrete object by the geometrical figure in terms of a continuous geometry (Figure 1c) and the construction of the skeleton for this figure (Figure 1d). The resulting skeleton is considered as a continuous skeleton of discrete objects.

Both approaches have their advantages and disadvantages.

The main advantage of the discrete approach is the simplicity of the algorithm and a graphic visualization of the skeleton in the source raster format (Figure 1a,b).

A continuous approach has its advantages. The main advantage of the approach is the continuous medial representation of the object's shape [4] as a geometrical graph with a radial function, which determines the width of the object. Radial function sets at each point of the geometrical graph radius of the inscribed circle centered at this point. Continuous medial representation allows the use of graph theory and computational geometry algorithms for image shape analysis and recognition.

It is well known that the topological structure of the skeleton is very sensitive to any changes in the boundaries of shape, in particular, to noise in the image. Therefore, all skeletonization algorithms, both discrete and continuous need special post-processing (pruning) to eliminate the effects of noise in the

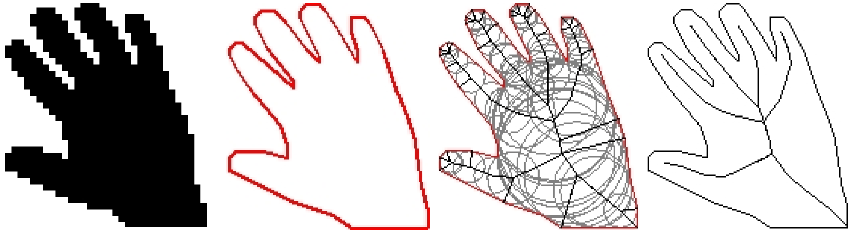

Figure 2: (a) – source binary image, (b) – polygonal approximation, (c) – medial representation of the polygon, (d) – resultant skeleton after pruning.

resulting skeleton. Pruning discrete skeleton is based on heuristic criteria. But pruning of a continuous skeleton is based on a rigorous formal criteria and methods.

Another advantage of the continuous approach is its higher computational efficiency. Experiments show that the computation of the continuous skeleton is significantly lower than the discrete skeleton, even taking into account the fact that the discrete approach is more suited to parallel computing.

Comparative analysis shows the advantages of continuous skeleton compared to discrete. These advantages are mathematical rigor, information content and computational efficiency. But to realize these benefits, we need a continuous model of a binary image, which allows us to apply well-developed ideas of computational geometry to skeletonization. We focus on the development and use of this model.

We offer the full implementation of the continuous approach to the skeleton construction for binary raster images of any complexity [3]. The author thanks the Russian Foundation for Basic Research for the support on this study.

The proposed idea consists of three parts (Figure 2):

- Approximation of binary images (Figure 2a) by polygonal figures (Figure 2b);
- Construction of Voronoi diagram of obtained set of figures by methods of computational geometry (Figure 2c);
- Obtaining the skeleton from the Voronoi diagram in a convenient format for further analysis (Figure 2d). The choice of a polygonal figure for the approximation is explained by the fact that the skeleton of a polygonal figure is a fairly simple structure and can be obtained from the Voronoi diagram of line segments of this figure. There are well developed methods

of computational geometry that can be used for the construction of such a diagram. On the other hand, using a polygonal approximation raises the question of removing "noisy" branches of the skeleton.

The implementation of this approach required solving of several complex problems.

1. The correct approximation of the binary bitmap by polygonal figures. By correctness we mean the construction of such figures, whose boundaries are described by simple polygons that do not intersect each other and have no self-intersections.

2. Obtaining Voronoi diagrams for polygonal figures. It is possible to use known algorithms for this purpose. However, these algorithms work effectively with a simply-connected figures (without holes). But for complex scenes consisting of a large number of multiply connected elements, they require too much time for computation. Our proposed algorithm is effective for just such scenes. It is based on the construction of the dual graph of Voronoi diagram of line segments, the so-called Delaunay graph.

3. Regularization of the skeleton based on the pruning. Regularization is necessary because the approximation of the bitmap image by polygonal figures with a given accuracy is not unique. But the skeleton that we want should be invariant to this approximation. Therefore, we must distinguish a normal part of the Voronoi diagram of a polygonal shape, which in some form is present in any variant of approximation. This problem occurs in both discrete and continuous approach. It is solved by cutting of some skeleton branches. A continuous approach allows us to perform this cutting on the basis of a mathematically rigorous formal criterion. The resulting subgraph of the Voronoi diagram is called the skeleton base.

4. The transformation of the skeleton in the format of the compound Bezier curve. The skeleton of a polygonal figure is a geometrical graph whose edges are described by straight line segments and quadratic parabola. The data structure describing a graph is very complicated. The proposed method for representing parabolic segments of the skeleton graph by quadratic Bezier curves allows us to simplify this data structure. As a result, the skeleton is described by means of a planar rectilinear graph. The general scheme of the algorithm that implements the approach described is shown in Figure 3.

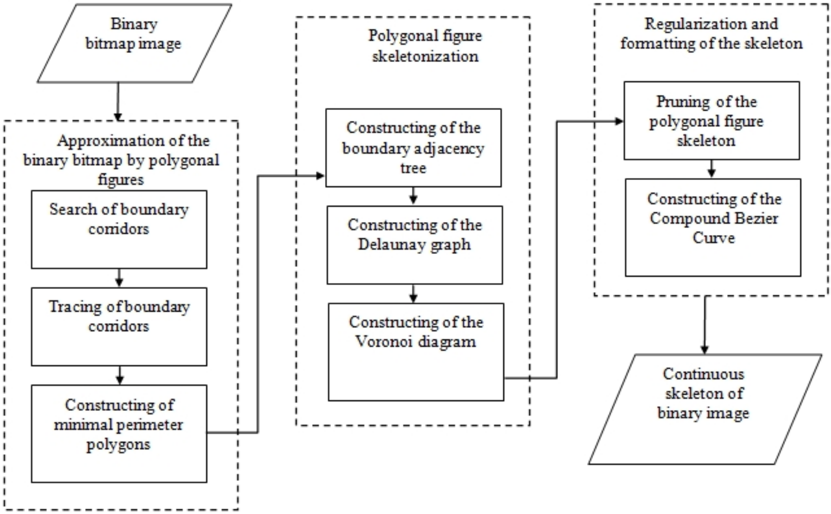

Figure 3: The structure of the algorithm for constructing a continuous skeleton of binary image bitmap.

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Homological persistence of maps

Marian Mrozek*

Abstract

We study the homological persistence of self-maps.

When a topological space is known only from sampling, persistence provides a useful tool to study its homological properties. In many applications one can sample not only the space, but also a map acting on the space. The understanding of the topological features of the map is often of interest, in particular in time series analysis. We consider the concept of persistence in finite dimensional vector spaces and combine it with a graph approach to computing homology of maps in order to study the persistence of eigenspaces of maps induced in homology.

This is research in progress, joint with Herbert Edelsbrunner and Grzegorz Jabłoński.

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On homothetic copy of a simplex which contains a convex body

Mikhail Nevskii*

Abstract

Let C be a convex body and let S be a nondegenerate simplex in $\mathbb{R}^n$. We examine the minimal $\sigma > 0$ such that a translate of σS contains C.

Let $C \subset \mathbb{R}^n$ be a convex body, i.e., compact convex set which has nonempty interior. By σC we mean the homothetic copy of C with center of homothety in the center of gravity of C and coefficient σ. Denote by $d_i(C)$ the ith axial diameter of C, i.e., the length of a longest segment in C parallel to the ith coordinate axis. The term *axial diameter* was introduced by Scott [1, 2].

Suppose C_1 and C_2 are the convex bodies. The symbol $\alpha(C_1; C_2)$ denotes the minimal $\sigma > 0$ such that C_1 is contained in a translate of σC_2. By definition, put $Q_n = [0,1]^n$. The result of Scott (see [1; theorem 1]) implies the following statement. *If we can inscribe a homothetic copy of Q_n into C, then*

$$\alpha(Q_n; C) \leqslant \sum_{i=1}^{n} \frac{1}{d_i(C)}. \tag{1}$$

This inequality holds true for each convex body C (see [3]). Moreover, if C is a simplex, then (1) becomes an equality. In other words, for each n-dimensional nondegenerate simplex S

$$\alpha(Q_n; S) = \sum_{i=1}^{n} \frac{1}{d_i(S)} \tag{2}$$

(see [4; theorem 4]). The aim of the present paper is to extend (2) to an arbitrary convex body C instead of Q_n.

Let S be a nondegenerate simplex in $\mathbb{R}^n$. Denote by $\lambda_1(x) \dots, \lambda_{n+1}(x)$ the barycentric coordinates of $x \in \mathbb{R}^n$ with respect to S. It is well-known that $\lambda_j(x)$ are the polynomials of degree 1 (or linear functions) in x (see [3] for detailes).

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In the sequel $x^{(j)} = \left(x_1^{(j)}, \ldots, x_n^{(j)}\right)$ $(1 \leqslant j \leqslant n)$ denote the vertices of simplex S. Let $\mathbf{A}^{-1} = (l_{ij})$ be the inverse matrix for the matrix

$$\mathbf{A} := \begin{pmatrix} x_1^{(1)} & \ldots & x_n^{(1)} & 1 \\ x_1^{(2)} & \ldots & x_n^{(2)} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{(n+1)} & \ldots & x_n^{(n+1)} & 1 \end{pmatrix}.$$

Then we have $\lambda_j(x) = l_{1j}x_1 + \ldots + l_{nj}x_n + l_{n+1,j}$.

Theorem 1. *For every convex body C,*

$$\alpha(C;S) = \sum_{j=1}^{n+1} \max_{x\in C}(-\lambda_j(x)) + 1. \tag{3}$$

In the case $C = Q_n$ (3) is equivalent to the equalities

$$\alpha(Q_n;S) = \sum_{i=1}^{n} \frac{1}{d_i(S)} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n+1} |l_{ij}|.$$

Define $\beta(C;S)$ as the greatest σ such that S contains a translate of σC. It is evident that $\beta := \beta(C;S) = \alpha(C;S)^{-1}$. Denote by C' a unique translate of βS contained in S. In other words, C' is the maximum homothetic copy of C contained in S. We note the following way to compute C'.

Theorem 2. *Let z be an arbitrary point of C. By definition, put*

$$\mu_j := \max_{x\in C}(\lambda_j(z) - \lambda_j(x)),$$

$$\tau_j := \frac{\mu_j}{\sum_{k=1}^{n+1}\mu_k}; \quad z^* := \sum_{j=1}^{n+1} \tau_j x^{(j)}.$$

Then we have

$$\beta = \left(\sum_{j=1}^{n+1} \mu_j\right)^{-1},$$

$$C' = \{y \in \mathbb{R}^n : y = z^* + \beta(x - z),\, x \in C\}.$$

The righthand set of the latter equality remains the same for all $z \in C$.

With the use of theorem 2 we immediately obtain the following description of the minimal positive copy $\tilde{S}$ of S which contains C. For brevity put $\alpha := \alpha(C; S)$.

Theorem 3. *A unique translate of αS which contains C has the form*

$$\widetilde{S} = \{x \in \mathbb{R}^n : x = z + \alpha(y - z^*), \, y \in S\}.$$

Here z and z^* are the points from the condition of theorem 2.

Theorems 1–3 are proved in [5]. Also [5] includes the examples corresponding to the special cases when C is a parallelotope or a simplex.

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Vertices of the cut polytope relaxations and 3-uniform hypergraphs

Andrey Nikolaev*

Abstract

We study a relationship between a special class of hypergraphs and properties of the relaxation points $M_{n,k}$ of a cut polytope. We prove that, for sufficiently large n, the polytopes $M_{n,4}$ and $M_{n,5}$ have points whose decompositions in the vertices of $M_{n,3}$ contain no integer vertices.

A hypergraph – is a generalized form of a graph in which edges can connect not only two vertices, but any subset of vertices [1]. A hypergraph is called k-uniform, if all its hyperedges have size exactly k.

Consider the set of 3-uniform mixed hypergraphs of the form $G = (V, E, A)$, where V is the vertex set, i.e., $V = \mathbb{N}_n = \{1, 2, 3, \dots, n\}$; E is the set of unoriented edges, i.e., $E = \{(i, j, k)\} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$; and A is the set of oriented edges, i.e., $A = \{((i, j), k)\} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where the pair of vertices (i, j) is the beginning of the edge and the vertex k is the end of the edge.

We introduce the operation of inversion of the i-th vertex in a hypergraph $G = (V, E, A)$ which transforms all edges incident to this vertex, as follows:

$$\begin{gathered}(i,j,k) \to ((j,k),i),\\ ((j,k),i) \to (i,j,k),\\ ((i,j),k) \to ((i,k),j).\end{gathered}$$

The result of applying the inversion operation is a new 3-uniform mixed hypergraph $G^{'} = Inv_i(G) = (V, E^{'}, A^{'})$.

Let G_I denote the class of hypergraphs $G = (V, E, A)$ for which the set E of unoriented edges is nonempty and remains nonempty under all possible inversions.

$$G = (V, E, A) \in G_I \Leftrightarrow \begin{cases} E \neq \oslash, \\ \forall W \subseteq V: \ G^{'} = Inv_W(G) = (V, E^{'}, A^{'}), \\ \hfill \text{where } E^{'} \neq \oslash. \end{cases}$$

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Thus, the class of hypergraphs G_I is closed under the operation of vertices inversion:

$$G = (V, E, A) \in G_I \;\Rightarrow\; \forall W \subseteq V : Inv_W(G) \in G_I.$$

Consider the following restriction of the 3-satisfiability problem (3-SAT), which is known as the monotone 3-satisfiability problem for different literals (monotone not-all-equal 3-SAT, or MNAE 3-SAT) [2].

Given. A set $U = \{u_1, ..., u_n\}$ of logical variables and a set of ternary disjunctions $C = \{c_j = u_{j_1} \vee u_{j_2} \vee u_{j_3}; j \in \mathbb{N}_n\}$.

Question. Does there exist a satisfying set of truth assignments for C such that each disjunction in C contains at least one true and at least one false literal?

To this problem the general 3-SAT problem reduces [3] (see also [2]); therefore, the MNAE 3-SAT problem is NP-complete.

With each individual problem $Z \in$ MNAE 3-SAT we associate a hypergraph $G(Z) = (V, E, A)$ of the form specified above (we call it the hypergraph of problem Z) according to the following rules:

1. $|V| = |U| = n$;

2. three vertices i, j, k of the hypergraph $G(Z)$ form an unoriented edge $(i, j, k) \in E$ if and only if the logical variables u_i, u_j and u_k are contained in the same disjunction from C;

3. $A = \oslash$.

It is easy to show that, for an individual problem $Z \in$ MNAE 3-SAT, the answer is "no" if and only if the associated hypergraph $G(Z)$ belongs to the class G_I. Thus, the following theorem is valid.

Theorem 1. *The recognition problem of the form "Is it true that a hypergraph G does not belong to the class G_I?" is NP-complete.*

Note that Theorem 1 is formulated in a negative way: as a recognition of not belonging to the class G_I because, strictly speaking, exactly to this problem MNAE 3-SAT can be reduced.

The recognition of hypergraph belonging to the class G_I can also be viewed as a generalized version of the known NP-complete problem of coloring 3-uniform hypergraph into 2 colors: "Is it possible to color the vertices of a 3-uniform hypergraph into 2 colors such that no three adjacent vertices share the same color (are not monochrome)".

In what follows, we use hypergraphs of the form specified above to describe properties of relaxation points of cut polytopes.

In [4], the class of polytopes $M_n \subseteq \mathbb{R}^{4n^2}$, $n \in \mathbb{N}$ was defined; polytopes from this class are now known as rooted semimetric polytopes [5] The linear constraints specifying M_n have the form

$$x_{i,j} + y_{i,j} + z_{i,j} + t_{i,j} = 1, \tag{26}$$

$$x_{i,j} + y_{i,j} = x_{k,j} + y_{k,j}, \tag{27}$$

$$x_{i,j} + z_{i,j} = x_{i,l} + z_{i,l}, \tag{28}$$

$$x_{i,j} = x_{j,i},\ t_{i,j} = t_{j,i},\ y_{i,j} = z_{j,i}, \tag{29}$$

$$y_{i,i} = z_{i,i} = 0, \tag{30}$$

$$x_{i,j} \geqslant 0,\ y_{i,j} \geqslant 0,\ z_{i,j} \geqslant 0,\ t_{i,j} \geqslant 0, \tag{31}$$

where i, j, k, l range independently over $1, ..., n$.

Polytopes from this class have a number of special features, which provoke significant interest in such polytopes (see [5, 6, 7]). In particular, in [8] the polynomial solvability of the following problem was proved: Given a linear objective function, determine whether $\max\{f(u) : u \in M_n\}$ is attained at an integer vertex of the polytope M_n (this problem is known as the integerness recognition problem).

The polytope M_n^Z, generated by all integer vertices of M_n is called the cut polytope, because the well-known NP-complete problem of maximal cut (as well as a number of other problems) reduces to optimizing a linear objective function on M_n^Z. Therefore, M_n is a relaxation polytope for the cut problem, or a relaxation of the cut polytope.

Let us define, following [5], higher level relaxations. For this purpose, we choose a positive integer k ($k < n$) and consider a system S of inequalities determining the polytope M_n^Z; we denote the number of these inequalities by Θ. For each k-element subset $\nu = \{\nu_1, \nu_2, \ldots, \nu_k\}$ of $\mathbb{N}_n$, we consider the system S_ν, obtained from S by replacing the variables $x_{i,j}$, $y_{i,j}$, $z_{i,j}$ and $t_{i,j}$, by x_{ν_i,ν_j}, y_{ν_i,ν_j}, z_{ν_i,ν_j} and t_{ν_i,ν_j}, respectively. Then, we augment system (1)–(6) by the $\Theta \cdot C_n^k$ resulting inequalities and let $M_{n,k}$ denote the polytope determined by the augmented system of constraints.

Polytopes M_1 and M_2 does not have noninteger vertices and, obviously, $M_{n,1} = M_{n,2} = M_n$. Thus, $M_{n,3}$ is the first relaxation of the cut polytope different from M_n. The relaxation $M_{n,3}$ is determined by system (1)–(6) and by the additional constraints

$$x_{i,j} + t_{i,j} + x_{i,k} + t_{i,k} + y_{j,k} + z_{j,k} \leqslant 2, \tag{32}$$

$$x_{i,j} + t_{i,j} + y_{i,k} + z_{i,k} + x_{j,k} + t_{j,k} \leqslant 2, \tag{33}$$

$$y_{i,j} + z_{i,j} + x_{i,k} + t_{i,k} + x_{j,k} + t_{j,k} \leqslant 2, \tag{34}$$

$$y_{i,j} + z_{i,j} + y_{i,k} + z_{i,k} + y_{j,k} + z_{j,k} \leqslant 2, \tag{35}$$

for each triple $i, j, k \in \mathbb{N}_n$, where $i < j < k$ [7, 8].

Consider an individual problem $Z \in$ MNAE 3-SAT, where $U = \{u_1, u_2, \ldots, u_n\}$ and the set C contains p disjunctions, and the polytope $M_{n,3}$. Let us construct the objective function

$$\forall x \in \mathbb{R}^{4n^2} : \ f(x) = \sum_{i,j,k} (y_{i,j} + z_{i,j} + y_{i,k} + z_{i,k} + y_{j,k} + z_{j,k})$$

for all triples i, j, k such that the logical variables u_i, u_j and u_k are contained in a common disjunction from C.

Obviously, we have $\max\{f(x) : \ x \in M_{n,3}\} = 2p$ on the polytope $M_{n,3}$, and if this maximum is attained at an integer vertex, then the set of disjunctions corresponding to the problem under consideration is satisfiable; otherwise, it is not satisfiable.

Thus, the MNAE 3-SAT problem reduces to the integerness recognition problem on $M_{n,3}$.

The proof of the abovementioned polynomial solvability of the integerness recognition problem on M_n [8] is based on following statement.

Theorem 2 (Vladimir Bondarenko, Boris Uryvaev, [8])**.** *Each point of the polytope $M_{n,3}$ is a convex combination of vertices of the polytope M_n, among which there is at least one integer vertex.*

In what follows, we show that the situation with subsequent relaxations is fundamentally different.

Note that each point $u \in M_{n,3}$ can be assigned a 3-uniform mixed hypergraph of the form specified above, which we call the hypergraph of the point $G(u)$ according to the following rules:

1. $V = N_n$;
2. $(i, j, k) \in E(u)$ if and only if $y_{i,j} + z_{i,j} + y_{i,k} + z_{i,k} + y_{j,k} + z_{j,k} = 2$;
3. $((i, j), k) \in A(u)$ if and only if $y_{i,j} + z_{i,j} + x_{i,k} + t_{i,k} + x_{j,k} + t_{j,k} = 2$.

Theorem 3. *If the hypergraph $G(u)$ of some point $u \in M_{n,3}$ belongs to the class G_I, then any decomposition of u in a convex combination of vertices of $M_{n,3}$ contains no integer vertices.*

The converse is generally not true. There exist points in the polytope $M_{n,3}$ (including those different from noninteger vertices) and even points in the polytope $M_{n,4}$ such that all decompositions of these points in a convex combinations of $M_{n,3}$ vertices contain no integer vertices, but their hypergraphs do not belong to the class G_I. A necessary and sufficient condition can be formulated as follows

Theorem 4. *Any decomposition of u in a convex combination of vertices of $M_{n,3}$ contains no integer vertices if and only if point u satisfies the following conditions:*

$$\forall v = Inv_U(u)\ (U \subseteq \mathbb{N}_n),\ \exists i, j, k \in \mathbb{N}_n :$$

$$x_{i,j} = 0 \quad or \quad y_{i,j} + z_{i,j} + y_{i,k} + z_{i,k} + y_{j,k} + z_{j,k} = 2.$$

Nevertheless, the more simple sufficient condition is more effective at finding specific points of the cut polytope relaxations $M_{n,k}$, any decomposition of which in a convex combination of the $M_{n,3}$ vertices contains no integer vertices. Based on the Theorem 3 was proved the following

Theorem 5. *For any $n \geqslant 5$ and $q \geqslant 195$, there exist points u in the polytope $M_{n,4}$ and v in the polytope $M_{q,5}$ whose hypergraphs $G(u)$ and $G(v)$ belong to the class G_I.*

Question with the subsequent relaxations of the cut polytope is still open, however, can be expected to take place the following

Conjecture 1. For any positive integer k there exists a number n, that in the $M_{n,k}$ polytope there are points which hypergraphs belong to the class G_I.

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Oriented graphs and complex spherical codes

Hiroshi Nozaki *

Abstract

Let X be a finite set on the d-dimensional complex unit sphere. The set X is called a complex spherical s-code if the set of the usual Hermitian inner products of two distinct vectors in X has size s. A complex spherical 2 or 3-code naturally has the structure of an oriented graph. In this talk, we introduce the way to embed from a given oriented graph to a minimal dimensional complex sphere as a 2 or 3-code having the graph structure. Moreover we classify or characterize the largest possible 2- or 3-codes for some dimensions. This includes a characterization of skew-Hadamard matrices, and some classification of tight complex spherical designs. This research is a joint work with Sho Suda.

Let X be a finite set on the d-dimensional complex unit sphere $\Omega(d)$. The set X is called a *complex spherical s-code* if the set of the usual Hermitian inner products of two distinct vectors in X has size s. The value s is called the *degree* of X. For two subsets of $\Omega(d)$, we say that they are *isomorphic* if there exists a unitary transformation from one to the other. One of the basic problems on complex spherical s-codes is to classify the largest possible s-codes for fixed s and d up to isomorphism. The s-code which has the largest possible cardinality is said to be *largest*.

A similar concept s-distance set is known for Euclidean finite sets. An s-distance set is a finite set in the Euclidean space $\mathbb{R}^d$ which has only s Euclidean distances between distinct points. The largest s-distance sets are studied in several papers (see [2, 8, 10, 12, 13, 14, 15, 19]). A spherical s-distance set particularly deserves an attention because of the relationship to association schemes and spherical t-designs. Delsarte, Goethals, and Seidel established one of fundamental theories for spherical algebraic combinatorics in their paper [10]. The remarkable result is that a spherical set satisfying $t \geqslant 2s-2$ has the structure of a certain special symmetric association scheme, which has the Q-polynomial property. Recently Roy and Suda [17] defined the concept of complex spherical designs, and gave the complex spherical analogue of the Euclidean theory in [10] between complex spherical designs, some codes including s-codes, and non-symmetric association schemes.

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Roy and Suda [17] gave a certain natural upper bound for the cardinality of complex spherical s-codes, which is called an absolute bound. A complex spherical s-code is said to be *tight* if it attains the absolute bound. A tight complex spherical s-code is most important concept in this research, because it becomes a minimal complex spherical design, and has the structure of a commutative association scheme [17]. Euclidean spherical distance sets have a similar situation, and a lot of researchers tried to classify tight distance sets [3, 4, 5, 6, 9, 10, 16]. Indeed a tight spherical s-distance set is classified except for $s = 2$. Tight distance sets, namely tight spherical designs are closely related to various good spherical configurations, for instance optimal codes, or kissing number configurations [1, 7, 10]. In the present talk, we introduce that the existence of tight complex spherical 2-codes is equivalent to that of skew Hadamard matrices. The number of non-isomorphic tight complex spherical 2-codes in $\mathbb{C}^d$ is calculated for $d \leqslant 14$:

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\lvert X \rvert$	3	4	7	8	11	12	15	16	19	20	23	24	27	28
#	1	2	1	4	1	8	2	240	2	8956	37	11339044	722	9897616700

Table 2: Largest complex 2-code X in $\Omega(d)$

Moreover tight complex spherical 3-codes are completely classified for any dimension, actually tight 3-codes in $\mathbb{C}^d$ exist only for $d = 1, 2$.

A Euclidean 2-distance set has the structure of a simple undirected graph by relating the short distance to the edge. Actually Einhorn–Schoenberg [11] showed the way to embed from the vertex set of a given graph to a minimal-dimensional Euclidean space. The classification of larger distance sets is possible from the finite number of simple graphs, and this especially progresses the research on the largest Euclidean 2-distance sets [13]. Later Roy [18] determined the minimal dimension by some values about the eigenspaces of the adjacency matrix of the graph. One of purposes of this research is to make the complex spherical analogue of these results in [11, 13, 18], and classify the largest 2- or 3-codes in $\Omega(d)$.

We can observe a complex spherical 2- or 3-code has the structure of an oriented graph, that is a digraph which has no symmetric pair of directed edges. In particular, a 2-code has the structure of a complete oriented graph so called a tournament. In this talk, we introduce the minimal dimension of the complex spheres which contain 2-codes obtained from a given tournament. This is the analogue of Roy's result for complex spherical 2-codes. For complex spherical 3-codes, it is hard to obtain a similar result to Roy's. However we can give a certain algorithm to classify the largest 3-codes from oriented graphs. By a current computer, we can classify the largest 3-codes for dimension $d \leqslant 3$:

d	1	2	3
$\lvert X\rvert$	4	8	9
#	1	1	35

Table 2: Largest complex 3-code X in $\Omega(d)$

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ε-Samples for Kernels

Jeff M. Phillips*

We study the L_∞ error in kernel density estimates of points sets by a kernel density estimate of their subset. Formally, we start with a size n point set $P \subset \mathbb{R}^d$ and a kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Then a kernel density estimate KDE_P of a point set P is a convolution of that point set with a kernel, defined at any point $x \in \mathbb{R}^d$:

$$\text{KDE}_P(x) = \sum_{p \in P} \frac{K(x,p)}{|P|}.$$

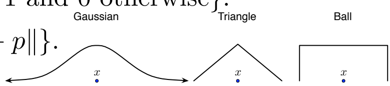
The goal is to construct a subset $S \subset P$, and bound its size, so that it has ε-bounded L_∞ error, i.e.

$$L_\infty\left(\text{KDE}_P, \text{KDE}_S\right) = \max_{x \in \mathbb{R}^d} \left|\text{KDE}_P(x) - \text{KDE}_S(x)\right| \leqslant \varepsilon.$$

We call such a subset S an *ε-sample of a kernel range space* $(P, \mathcal{K})$, where $\mathcal{K}$ is the set of all functions $K(x, \cdot)$ represented by a fixed kernel K and an arbitrary center point $x \in \mathbb{R}^d$. Our main result is the construction in $\mathbb{R}^2$ of an ε-sample of size $O((1/\varepsilon)\sqrt{\log(1/\varepsilon)})$ for a broad variety of kernel range spaces.

We will study this result through the perspective of three types of kernels. We use as examples the ball kernels $\mathcal{B}$, the triangle kernels $\mathcal{T}$, and the Gaussian kernels $\mathcal{G}$; we normalize all kernels so $K(p,p) = 1$.

- For $K \in \mathcal{B} : K(x,p) = \{1 \text{ if } \|x-p\| \leqslant 1 \text{ and } 0 \text{ otherwise}\}$.
- For $K \in \mathcal{T} : K(x,p) = \max\{0, 1 - \|x-p\|\}$.
- For $K \in \mathcal{G} : K(x,p) = \exp(-\|x-p\|^2)$.



Our main result holds for $\mathcal{T}$ and $\mathcal{G}$, but not $\mathcal{B}$. However, in the context of combinatorial geometry, kernels related to $\mathcal{B}$ (binary ranges) seem to have been studied the most heavily from an L_∞ error perspective, and require larger ε-samples. We also show a lower-bound that such a result cannot hold for $\mathcal{B}$.

We re-describe this result next by adapting (binary) range spaces and discrepancy; these same notions will be used to prove our result.

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Range spaces. A kernel range space is an extension of the combinatorial concept of a range space. Let $P \subset \mathbb{R}^d$ be a set of n points. Let $\mathcal{A} \subset 2^P$ be the set of subsets of P, for instance when $\mathcal{A} = \mathcal{B}$ they are defined by containment in a ball. The pair $(P, \mathcal{A})$ is called a *range space*.

Thus we can re-imagine a kernel range space $(P, \mathcal{K})$ as the family of *fractional* subsets of P, that is, each $p \in P$ does not need to be completely in (1) or not in (0) a range, but can be fractionally in a range described by a value in $[0, 1]$. In the case of the ball kernel $K(x, \cdot) \in \mathcal{B}$ we say the associated range space is a *binary range space* since all points have a binary value associated with each range, corresponding with *in* or *not in*.

Colorings and discrepancy. Let $\chi : P \to \{-1, +1\}$ be a *coloring* of P. The *combinatorial discrepancy* of $(P, \mathcal{A})$, given a coloring χ is defined $d_\chi(P, \mathcal{A}) = \max_{R \in \mathcal{A}} |\sum_{p \in R} \chi(p)|$. For a kernel range space $(P, \mathcal{K})$, this is generalized as the *kernel discrepancy*, defined $d_\chi(P, \mathcal{K}) = \max_{x \in \mathbb{R}^d} \sum_{p \in P} \chi(p) K(x, p)$; we can also write $d_\chi(P, K_x) = \sum_{p \in P} \chi(p) K(x, p)$ for a specific kernel K_x, often the subscript x is dropped when it is apparent. Then the *minimum kernel discrepancy* of a kernel range space is defined $d(P, \mathcal{K}) = \min_\chi d_\chi(P, \mathcal{K})$. See Matouśek's [11] and Chazelle's [4] books for a masterful treatments of this field when restricted to combinatorial discrepancy.

Constructing ε-samples. Given a (binary) range space $(P, \mathcal{A})$ an *ε-sample* (a.k.a. an ε-approximation) is a subset $S \subset P$ such that the density of P is approximated with respect to $\mathcal{A}$ so

$$\max_{R \in \mathcal{A}} \left| \frac{|R \cap P|}{|P|} - \frac{|R \cap S|}{|S|} \right| \leqslant \varepsilon.$$

Clearly, an ε-sample of a kernel range space is a direct generalization of the above defined ε-sample for (binary) range space. In fact, recently Joshi *et.al.* [8] showed that for any kernel range space $(P, \mathcal{K})$ where all super-level sets of kernels are described by elements of a binary range space $(P, \mathcal{A})$, then an ε-sample of $(P, \mathcal{A})$ is also an ε-sample of $(P, \mathcal{K})$. For instance, super-level sets of $\mathcal{G}, \mathcal{T}$ are balls in $\mathcal{B}$.

ε-Samples are a very common and powerful coreset for approximating P; the set S can be used as proxy for P in many diverse applications (c.f. [1, 12, 6]). For binary range spaces with constant VC-dimension [14] a random sample S of size $O((1/\varepsilon^2) \log(1/\delta))$ provides an ε-sample with probability at least $1 - \delta$ [9]. Better bounds can be achieved through deterministic approaches as outlined by Chazelle and Matousek [5], or see either of their books for more details [4, 11]. This approach is based on the following rough idea. Construct a low discrepancy coloring χ of P, and remove all points $p \in P$ such that $\chi(p) = -1$. Then repeat these color-remove steps until only a small number of points are left (that are always colored $+1$) and not too much error has accrued. As such, the best bounds for the size of ε-samples are tied directly to discrepancy. As spelled out explicitly by Phillips [12] (see also [11, 4] for more classic references), for a range space $(P, \mathcal{A})$ with discrepancy $O(\log^\tau |P|)$ (resp. $O(|P|^\psi \log^\tau |P|)$) that can be constructed in time $O(|P|^w \log^\phi(|P|))$, there is an ε-sample of size $g(\varepsilon) = O((1/\varepsilon)\log^\tau(1/\varepsilon))$ (resp. $O(((1/\varepsilon)\log^\tau(1/\varepsilon))^{1/(1-\psi)})$) that can be constructed in time $O(w^{w-1} n \cdot (g(\varepsilon))^{w-1} \cdot \log^\phi(g(\varepsilon)) + g(\varepsilon))$. Although, originally intended for binary range spaces, these results hold directly for kernel range spaces.

Our Results

Our main structural result is an algorithm for constructing a low-discrepancy coloring χ of a kernel range space. The algorithm is relatively simple; we construct a min-cost matching of the points (minimizes sum of distances), and for each pair of points in the matching we color one point $+1$ and the other -1 at random.

Theorem 1. *For $P \subset \mathbb{R}^d$ of size n, the above coloring χ, has discrepancy $d_\chi(P, \mathcal{T}) = O(n^{1/2-1/d}\sqrt{\log(n/\delta)})$ and $d_\chi(P, \mathcal{G}) = O(n^{1/2-1/d}\sqrt{\log(n/\delta)})$ with probability at least $1 - \delta$.*

This implies an efficient algorithm for constructing small ε-samples of kernel range spaces.

Theorem 2. *For $P \subset \mathbb{R}^d$, with probability at least $1 - \delta$, we can construct in $O(n/\varepsilon^2)$ time an ε-sample of $(P, \mathcal{T})$ or $(P, \mathcal{G})$ of size $O((1/\varepsilon)^{2d/(d+2)} \log^{d/(d+2)}(1/\varepsilon\delta))$.*

Note that in $\mathbb{R}^2$, the size is $O((1/\varepsilon)\sqrt{\log(1/\varepsilon\delta)})$, near-linear in $1/\varepsilon$, and the runtime can be reduced to $O((n/\sqrt{\varepsilon})\log^5(1/\varepsilon))$. Furthermore, for $\mathcal{B}$, the best known upper bounds for discrepancy (which are tight up to a log factor) are noticeably larger at $O((1/\varepsilon)^{2d/(d+1)} \cdot \log^{d/(d+1)}(1/\varepsilon))$, especially in $\mathbb{R}^2$ at $O((1/\varepsilon)^{4/3}\log^{2/3}(1/\varepsilon))$.

We note that many combinatorial discrepancy results also use a matching where for each pair one is colored $+1$ and the other -1. However, these matchings are "with low crossing number" and are not easy to construct. For a long time these results were existential, relying on the pigeonhole principle. But, recently Bansal [2] provided a randomized constructive algorithm; (see also [10]). Yet still these are quite more involved than our min-cost matching. We believe that this simpler, and perhaps more natural, min-cost matching algorithm may be of independent practical interest.

Proof overview. The proof of Theorem 2 follows from Theorem 1, the above stated results in [12], and Edmond's $O(n^3)$ time algorithm for min-cost matching M [7]. So the main difficulty is proving Theorem 1. The full technical details can be found in [13]. Here we first outline the proof in $\mathbb{R}^2$ on $\mathcal{T}$. In particular, we focus on showing that the coloring χ derived from M, for any single kernel K, has $d_\chi(P, K) = O(\sqrt{\log(1/\delta)})$ with probability at least $1-\delta$. Then we extend this to an entire family of kernels with $d_\chi(P, \mathcal{K}) = O(\sqrt{\log(n/\delta)})$.

The key aspect of kernels required for the proof is a bound on their slope, and this is the critical difference between binary range spaces (e.g. $\mathcal{B}$) and kernel range spaces (e.g. $\mathcal{T}$). On the boundary of a binary range the slope is infinite, and thus small perturbations of P can lead to large changes in the contents of a range; all lower bounds for geometric range spaces seem to be inherently based on creating a lot of action along the boundaries of ranges. For a kernel $K(x,\cdot)$ with slope bounded by σ, we use a specific variant of a Chernoff bound that will depend only on $\sum_j \Delta_j^2$, where $\Delta_j = K(x, p_j) - K(x, q_j)$ for each edge $(p_j, q_j) \in M$. Note that $\sum_j |\Delta_j| \geqslant d_\chi(P, K)$ gives a bound on discrepancy, but analyzing this directly gives a $\mathsf{poly}(n)$ bound. Also, for a binary kernels $\sum_j \Delta_j^2 = \mathsf{poly}(n)$, but if the kernel slope is bounded then $\sum_j \Delta_j^2 \leqslant \sigma^2 \sum_j \|p_j - q_j\|^2$. Then we bound $\sum_j \|p_j - q_j\|^2 = O(1)$ within a constant radius ball, specifically the ball B_x for which $K(x,\cdot) > 0$. This follows (after some technical details) from a result of Bern and Eppstein [3].

Extending to $\mathcal{G}$ requires multiple invocations of the $\sum_j \Delta_j^2$ bound. Extending to $\mathbb{R}^d$ basically requires generalizing the matching result to depend on the sum of distances to the dth power and applying Jensen's inequality to relate $\sum_j \Delta_j^d$ to $\sum_j \Delta_j^2$.

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Additive structures in multiplicative subgroups

Ilya Shkredov*

Abstract

We investigate various "random" properties of multiplicative subgroups of finite fields such as intersections of such subgroups with its additive shifts, its basis properties, representations of subgroups as sumsets and so on. We give a survey of old and new results in the field.

Let p be a prime number, and $R \subseteq \mathbf{Z}_p^* = (\mathbf{Z}/p\mathbf{Z}) \setminus \{0\}$ be a multiplicative subgroup. Such subgroups were studied by various authors, see e.g. [1]–[14]. Most of the results show that the behavior of multiplicative subgroups similar to the behavior of randomly chosen with probability $|R|/p$ subset of $\mathbf{Z}_p^*$. In other words the characteristics of such subgroups are close to the correspondent properties of random sets. We consider the following characteristics

- intersections with additive shifts
- basis properties
- representations as sumsets.

Unfortunately, we have not deal with another important properties of multiplicative subgroups (although they demonstrate exactly the same behavior as random sets) such as its uniform distribution, estimates of its exponential sums, intersections with arithmetic progressions and others.

The question on intersection of multiplicative subgroups with its additive shift was considered firstly by A. Garcia and J.F. Voloch [11]. They proved, using deep algebraic ideas, that for any subgroup R, $|R| < (p-1)/((p-1)^{1/4}+1)$ and an arbitrary nonzero μ the following holds

$$|R \bigcap (R+\mu)| \le 4|R|^{2/3} \,. \tag{36}$$

D.R. Heath–Brown and S.V. Konyagin generalized (36) and gave another prove of the result in [8] (see also [9]). Their approach uses a well–known method of S.A. Stepanov [15]. We extend the result of Garcia–Voloch and also similar theorems from [8], [9] for the case of several additive shifts.

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Theorem 1 (I. D. Shkredov, I. V. V'ugin., [13]). *Let $R \subseteq \mathbf{Z}_p^*$ be a multiplicative subgroup, $k \ge 1$ be a positive integer, $|R| > k2^{2k+4}$. Let also $\mu_1, \dots, \mu_k$ be different nonzero residuals, and $Q = RQ$ be a R—invariant set, $0 \notin Q$, $|Q| < ((|R|/k)^{1/2k} - 1)^{2k+1}$, $p \ge 4k|R|(|Q|^{\frac{1}{2k+1}} + 1)$. Then*

$$\sum_{\lambda \in Q} |R \bigcap (R + \lambda \cdot \mu_1) \bigcap \dots \bigcap (R + \lambda \cdot \mu_k)| \le 4(k+1)(|Q|^{\frac{1}{2k+1}} + 1)^{k+1} |R| \,. \quad (37)$$

Now consider another additive characteristic of multiplicative subgroups, namely, the cardinality of their sums and differences. Bound (36) implies that (see [11])

$$|R \pm R| \gg |R|^{4/3} \quad (38)$$

for any subgroup R with $|R| \ll p^{3/4}$. D.R. Heath–Brown and S.V. Konyagin in [8] (see also [9]) proved

$$|R \pm R| \gg |R|^{3/2} \quad (39)$$

for all subgroups R such that $|R| \ll p^{2/3}$. Using a combinatorial idea, we improve inequality (39) in the following way

$$|R \pm R| \gg \frac{|R|^{5/3}}{\log^{1/2} |R|} \quad (40)$$

for subgroups R with the condition $|R| \ll p^{1/2}$. The inequalities above answer on a question from [5].

Bounds (38)–(40) are connected with the problem on basis properties of multiplicative subgroups, that is the question on finding the smallest integer l depending on the size of R such that $lR = \mathbf{Z}$, see e.g. [5]. There is an old conjecture that $\mathbf{Z}_p \subseteq R + R$ provided by $|R| > p^{1/2+\varepsilon}$. Let us formulate our result.

Theorem 2 (T. Schoen, I. D. Shkredov, [16]). *Let $R \subseteq \mathbf{Z}_p^*$ be a multiplicative subgroup such that $|R| \ge p^\kappa$, where $\kappa > \frac{99}{203}$. Then for all sufficiently large p we have $\mathbf{Z}_p^* \subseteq 6R$.*

The question on representations of the set of quadratic residuals R was considered in [17] (see also [18]).

Theorem 3 (A. Sárközy, [17]). *Let p be a prime number, R be the set of quadratic residuals. Suppose that $R = A + B$, $|A|, |B| \geq 2$. Then*

$$\frac{p^{1/2}}{3 \log p} < |A|, |B| < p^{1/2} \log p \,.$$

We obtain a series of results in the direction. For example, we reduce logarithmic factors in Theorem 3, prove that $R = A + A$ iff $p = 3$ and $A = \{2\}$, consider the case $R = A \dotplus A$, study almost sumsets and so on.

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On the number of complement regions in submanifolds arrangements

Igor Shnurnikov*

Abstract

We consider the arrangements of given number codimensional one submanifolds and study the sets of all possible connected components numbers.

Let us consider m–dimensional manifold M and the finite family $\{A_1, \dots, A_n\}$ of closed $(m-1)$–dimensional subsets. Let

$$f = |\pi_0(M \setminus \bigcup_{i=1}^{n} A_i)|$$

be the connected components number of the complement in M to the union of $A_1, \dots, A_n$. Let $F_n(M)$ be the set of numbers f for all possible arrangements of n subsets given type. The general question is to describe the sets $F_n(M)$ for arrangements of closed geodesics or totally geodesic surfaces in M. The sets $F_n(M)$ could be interesting in connection with Orlic and Solomon [1] statement that the region number in hyperplane arrangements equals to the cohomology ring dimension of the complement to complexified arrangement. N. Martinov [2] founded the sets of region numbers in real projective plane arrangements of lines and arrangements of pseudolines. Hence the sets of region numbers in standard sphere arrangements of big circles are also known.

Theorem 1. *Let us consider arrangements of $(m-1)$–dimensional flat subtori in the m–dimensional flat torus T^m, arrangements of closed (non–simple) geodesics in the flat Klein bottle KL^2, arrangements of closed simple geodesics in the surface R of the tetrahedron, arrangements of hyperplanes in the hyperbolic metric of Lobachevsky space L^m and finally hyperplane arrangements with*

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empty intersection of all hyperplanes in the real projective space P^m*. Then*

$$F_n(T^m) \supseteq \{n-m+1,\dots,n\} \cup \{l \in \mathbb{N} \mid l \geqslant 2(n-m)\} \tag{41}$$

$$F_n(KL^2) = \{n+1\} \cup F_n(T^2) \quad \textit{for } n \geqslant 2,\ F_1(KL^2) = \mathbb{N},$$

$$F_n(R) \subseteq \{n+1, 2n\} \cup \{l \in \mathbb{N} \mid l \geqslant 4n-6\} \quad \textit{for } n \geqslant 3, \tag{42}$$

$$F_n(L^m) \ = \ \left\{ \ f \in \mathbb{N} \ \mid \ n+1 \leqslant f \leqslant \sum_{i=0}^{m} \left(\begin{array}{c} n \\ i \end{array} \right) \ \right\}$$

first four numbers of $F_n(P^m)$ *for* $n \geqslant 2m+5$ *and* $m \geqslant 3$ *are:*

$$(n-m+1)2^{m-1}, 3(n-m)2^{m-2}, (3n-3m+1)2^{m-2}, 7(n-m)2^{m-3}.$$

Remark 1. The inclusion (41) turns into equality at least for two–dimensional tori. The inclusion (42) turns into equality iff all tetrahedron faces are equal acute–angled triangles.

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Sampling for Local Homology with Vietoris-Rips Complexes

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Recently, a multi-scale notions of local homology was proposed [1] to study the local structure of a stratified space around a given point from a point cloud sample. To give reconstruction guarantees, the approach relied on constructing embedded complexes which become difficult to construct in high dimensions. We derive sampling conditions under which the *persistence diagrams* used for estimating local homology, can be approximated using families of Vietoris-Rips complexes, whose simple construction is robust in any dimension. To the best of our knowledge, our results, for the first time, make stratification learning using local homology feasible in high-dimensions.

Advances in scientific and computational experiments have increased our ability to gather large collections of data points in high-dimensional spaces, far outpacing our capacity to analyze and understand them. For instance, in a large-scale simulation, one might want to understand the relationships between a large number of input parameters and their effects on a set of particular outcomes.

We approach the problem as follows, given a point cloud of data sampled from some underlying space, can we infer the topological structure of the space? Often we assume the support of the domain is either from a low-dimensional space with manifold structure, or more interestingly, contains mixed dimensionality and complexity. The former is a classic setting in *manifold learning.* The latter can often be described by a stratified set of manifolds and becomes a problem of particular interest in the field of *stratification learning.*

Stratified spaces, while not manifolds, can be decomposed into manifold pieces that are glued together in some uniform way. An important tool in stratification learning is the study of local spaces, that is, the neighborhoods surrounding singularities, where manifolds of different dimensionality and complexity intersect. We focus on sampling conditions for such neighborhoods, which allow us to begin examining how difficult certain reconstruction techniques are with respect to the geometric properties of the underlying shape. Our main task is to infer sampling conditions suitable for recovering local structures of stratified spaces, in particular, the *local homology groups*, from a possibly noisy sampled point set.

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We provide sampling conditions to recover the local structure of a stratified space from a point cloud sample, based on previously introduced [1] multi-scale notions of local homology. Our main results are:

- First, we extend previously introduced algebraic constructions [2] to the setting of local homology, for two multi-scale notions of local homology.
- For both notions of local homology, we approximate the persistence diagrams of the two relative homology filtrations with a filtration built on a set of sample points, and reduce the required algorithms to either the standard persistence algorithm or a simple variant.
- We show that we can use Vietoris-Rips complexes in our constructions. The simplicity and efficiency of building the Vietoris-Rips complexes in any dimension makes, for the first time, stratification learning based on local homology feasible in high-dimensions.
- We explore the potential applications of our results including stratification learning and computing well groups [3], a generalization of persistent homology designed to measure the robustness of maps to perturbations.

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Fast Algorithm for a Nearest Neighbor Search with Weak Metrics

Evgeniy A. Timofeev* Alexei Kaltchenko†

Abstract

A new fast algorithm is proposed for finding a k-nearest neighbor in a dictionary for a given input word. The algorithm has time complexity, which is independent of the dictionary size.

K-nearest neighbor search has been extensively used as a powerful non-parametric technique in numerous applications, including: pattern recognition, statistical classification, coding theory, computer vision, and DNA sequencing. However, as k-nearest neighbor search requires intensive computations, searching through the whole set is unacceptable particularly for a large set. Therefore, speeding-up the search is a key to make k-nearest neighbor search useful in practice.

Many similarity criterions (for example, the well-known Smith-Waterman distances between sequences or various distances between images) , although being widely used do not verify the triangular inequality.

It is clear (and known) that an appropriate choice of a metric can improve the search.

We introduce a wider class of so-called *weak* metrics, for which the triangle inequality holds with some constant $C > 1$. These new metrics have parameters which are non-decreasing functions.

We propose an algorithm for finding a k-nearest neighbor in a dictionary for a given input word. The algorithm's time complexity is independent of the size of the dictionary (but dependent of the given word length and the size of the alphabet).

We will apply this algorithm to the problem of entropy estimation [1, 2].

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1 Problem Statement

Let $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ be n words $\boldsymbol{x_i} = x_{i1} \ldots x_{im}$, $x_{ij} \in \mathcal{A}$, where $\mathcal{A}$ is a finite alphabet.

We define the following similarity on the space $\mathcal{A}^m$:

$$\begin{aligned} &\alpha^{(0)}(\boldsymbol{x}, \boldsymbol{y}) = 1; \\ &\alpha^{(m)}(a\boldsymbol{x}, b\boldsymbol{y}) = \begin{cases} \alpha^{(m-1)}(\boldsymbol{x}, \boldsymbol{y}) + 1, & a = b; \\ \lambda_{ab}(\alpha^{(m-1)}(\boldsymbol{x}, \boldsymbol{y})), & a \neq b; \end{cases} \end{aligned} \tag{43}$$

where $\lambda_{ab}(t) = \lambda_{ba}(t)$ are nondecreasing function such that $\lambda_{ab}(0) = 0$ and $\lambda_{ab}(t) \leqslant 1$, $0 \leqslant t < \infty$, $a \neq b \in \mathcal{A}$.

We impose the following restriction on the functions $\lambda_{ab}(t)$

$$\begin{aligned} &\forall a \in \mathcal{A} \;\; \exists \sigma_1(a), \ldots, \sigma_{A-1}(a), \;\; \{\sigma_1(a), \ldots, \sigma_{A-1}(a)\} = \mathcal{A} \setminus \{a\} \; : \\ &\lambda_{a\sigma_1(a)}(t_1) > \lambda_{a\sigma_2(a)}(t_2) > \cdots > \lambda_{a\sigma_{A-1}(a)}(t_{A-1}), \\ &\forall t_1, t_2, \ldots t_{A-1} > 0 \; : \; \lambda_{a\sigma_i(a)}(t_i) > 0. \end{aligned} \tag{44}$$

For a given word $\boldsymbol{X_0}$ and integer k, we want to find the following:

$$\max_{1 \leqslant i \leqslant n}{}^{(k)} \alpha^{(m)}(\boldsymbol{x_i}, \boldsymbol{X_0}),$$

where $\max^{(k)}\{X_1, \ldots, X_N\} = X_k$ if $X_1 \geqslant X_2 \geqslant \ldots \geqslant X_N$.

We stress that, using this similarity, we obtain a metric $\rho(\boldsymbol{x}, \boldsymbol{y}) = \theta^{-\alpha^{(\infty)}(\boldsymbol{x}, \boldsymbol{y})}$ on the space $\mathcal{A}^\infty$ of right-sided infinite sequences drawn from a finite alphabet $\mathcal{A}$, where $\theta > 1$.

Note also that the metric ρ with arbitrary functions $\lambda_{ab}(t)$ is bi-Lipschitz equivalent to the metric ρ with $\lambda_{ab}(t) \equiv 0$. Therefore, ρ is a weak metric (or near-metric), i.e. the triangle inequality holds with some constant $C > 1$.

2 Search Algorithm

2.1 Dictionary Representation

Dictionary ($\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$) will be stored as a trie T. The root is associated with the empty string.

We assign to each inner node i an auxiliary parameter ν_i such that ν_i is the number of descendant leaves in the subtrie with the root i.

Note that in our trie T all leaves are at level m.

Required memory is $\mathcal{O}(An)$.

2.2 Input

- The trie T with the auxiliary array ν,
- the given word $\boldsymbol{X_0}$ and integer k.

2.3 Algorithm Description

A pseudo-code description of the algorithm is presented below.

- Node V:=root.
- **for** j := 1 to m **do**

  1. Set $a := X_{0l}$. Let $c_1, \dots, c_s$ be all children of V such that c_i is the child with the edge label $\sigma_i(a)$.
  2. Find r such that

  $$z = \nu_{c_1} + \cdots + \nu_{c_{r-1}} < k, \quad z + \nu_{c_r} \geqslant k.$$

  3. Set $V := c_r$, $k := k - z$.

- The leaf V is the output result of the algorithm.

  If the word $\boldsymbol{x_j}$ corresponding to a leaf V, then

  $$\max_{1 \leqslant i \leqslant n}{}^{(k)} \alpha^{(m)}(\boldsymbol{x_i}, \boldsymbol{X_0}), = \alpha^{(m)}(\boldsymbol{x_j}, \boldsymbol{X_0}).$$

2.4 Algorithm Complexity

It is routine to verify that this algorithm is correct and requires $\mathcal{O}(mA))$ time.

The space complexity is $\mathcal{O}(An)$.

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