Triangulations and Cubulations of Manifolds: Research Statement

The author's paper [1] is devoted to the following problem: Given a set (with repetitions) of oriented $(n-1)$-dimensional combinatorial spheres $Y_1, \dots, Y_k$, decide whether there exists a simplicial (or cubical) n-dimensional oriented combinatorial manifold K whose set of links of vertices coincides up to an orientation-preserving isomorphism with the set $Y_1, \dots, Y_k$. If such combinatorial manifold exists we wish to construct it explicitly. The complete solution of this problem can hardly be obtained. In [1] the author has obtained certain partial positive results. The main result is the construction that, for any given set $Y_1, \dots, Y_k$ of oriented $(n-1)$-dimensional combinatorial spheres satisfying certain simple necessary condition, yields an oriented n-dimensional simplicial manifold K whose set of links of vertices coincides up to an isomorphism with the set

$$\underbrace{Y_1, \dots, Y_1}_{r}, \underbrace{Y_2, \dots, Y_2}_{r}, \dots, \underbrace{Y_k, \dots, Y_k}_{r}, Z_1, Z_2, \dots, Z_l, -Z_1, -Z_2, \dots, -Z_l$$

for some oriented $(n-1)$-dimensional combinatorial spheres $Z_1, Z_2, \dots, Z_l$, where $-Z$ is the sphere Z with the orientation reversed.

In the same paper the applications of this construction have been obtained
1) to cobordisms of manifolds with singularities,
2) to Steenrod's problem on realization of cycles,
3) to construction of local combinatorial formulae for the Pontryagin classes. (Explicit, though very ineffective formulae for all polynomials in rational Pontryagin classes have been constructed.)

Very close to this problem is the following problem that appeared in the paper of Cooper and Thurston [2] in 1988. Informally, the problem is to understand to which extent can be simplified the local combinatorial structure of a triangulation (or a cubulation) of the given manifold. More precise formulation is as follows. For every n, find a finite set of $(n-1)$-dimensional combinatorial spheres such that any n-dimensional PL manifold possesses a triangulation (or a cubulation) with links of all vertices isomorphic to some combinatorial spheres from the set constructed. Cooper and Thurston solved this problem for $n = 3$. Their result is as follows.

Theorem 1. *Any 3-dimensional manifold M^3 has a cubulation in which each edge is contained in 3, 4, or 5 cubes. Besides, the union of the edges each of which is contained in exactly 3 cubes and the union of the edges each of which is contained in exactly 5 cubes are the non-intersecting one-dimensional submanifolds in M^3.*

It follows that each 3-manifold can be triangulated with 3 types of links of vertices.

We have investigated the Cooper–Thurston problem for $n = 4$. Surprisingly, the answer depends on the signature of the manifold. For manifolds of zero signature a natural analogue of Theorem 1 has been obtained. For manifolds of non-zero signature two additional types of links of vertices are needed. Methods used in this problem are closely related to the methods used for construction of the manifolds with the prescribed sets of links of vertices in [1].

References

[1] Gaifullin A. A., *The construction of combinatorial manifolds with prescribed sets of links of vertices*, Izvestiya: Mathematics **72**:5 (2008), p. 845–899.

[2] Cooper D., Thurston W. P., *Triangulating 3-manifolds using 5 vertex link types.* Topology **27**:1 (1988), p. 23–25.