What Spaces Permit Fermat Points Construction and Melzak Algorithm?

Problem stated by Alexander Ivanov and Alexei Tuzhilin

Fermat Point, Shortest Trees, and $120°$-property. Classical Fermat Problem consists in finding in the Euclidean plane a point X minimizing the sum $f(X) = |A_1X| + |A_2X| + |A_3X|$ of distances from a three fixed points A_1, A_2, and A_3. The solution is referred as the *Fermat point*. The strict convexity of the distance function and, hence, the function f, implies the uniqueness of the Fermat point F for any triangle $\Delta = A_1A_2A_3$. If all the angles of the triangle Δ are less than $120°$, then F coincides with the Torricelli point, and otherwise F is the vertex of the largest angle. Recall that the Torricelli point T is the intersection point of three circles circumscribed around regular triangles $A_iB_jA_k$ located outside Δ on its sides A_iA_j, or as the intersection point of three segments A_iB_i. In this case T can be defined as the point such that all the angles A_iTA_j are equal to each other and, hence, equal to $120°$. The resulting *shortest tree* $G = \cup_i[A_iF]$ meets the $120°$*-property*: edges–segments meet each other by angle more or equal than $120°$.

PROBLEM. Investigate Fermat problem in sphere, in Lobachevski plane, in an Alexandrov surface of bounded curvature, in a normed plane. Is a Fermat point always unique? How to construct the set of all Fermat points?

Remarks. The $120°$-property remains valid in much more general situations: in Riemannian manifolds [1], in Alexandrov spaces of bounded curvature [2], in particular, in the surfaces of polyhedra. Therefore, in the sphere, in the Lobachevski plane or in an Alexandrov surface the Fermat point can be determined similarly as a point, which the sides of the triangle are seen by the angle more than or equal to $120°$, but it is quite unclear how to construct it geometrically. In a normed plane the uniqueness result is not valid.

Locally Minimal Networks and Melzak Algorithm. More general, there exists well-known Melzak algorithm that gives an opportunity to construct so-called *locally minimal trees* (planar trees consisting of straight segments meeting by angles more or equal than $120°$), see for example [3]. If we are given with a planar binary tree G, a finite subset M of the plane, referred as boundary, and the correspondence between M and the set of vertices of G with degree 1, then Melzak algorithm either construct locally minimal tree of type G joining M, or finds out that such a tree does not exist. At the first stage, we reduces M and G as follows: (a) change a pair of points m and m' from M such that the corresponding edges e and e' from G have a common vertex x by the third vertex w of the regular triangle $mm'w$ and (b) delete the edges e and e' from G and assign vertex x to the point w. As a result, we get new binary tree with new boundary set consisting of less number of points. At the end of the first stage we obtain a tree consisting of a single edge for the two points boundary. The corresponding locally minimal tree is the straight segment. On the second stage we start with this straight segment and reconstruct the tree constructed for smaller boundary to the tree for the bigger one. Each new vertex V of degree three can be constructed as an intersection of the tree with the circle circumscribed around one of the regular triangle from the first stage.

PROBLEM. Generalize the Melzak algorithm to the case of the Lobachevski plane, the sphere, an Alexandrov surface, a normed plane.

Remarks. In all the cases, it is not difficult to make a computer program calculating Fermat point or locally minimal tree for small number of boundary points numerically. We are interested in exact solutions important for theoretical constructions. Here even the case of small number of points is interesting. For example, it is not known how to determine which of three possible locally minimal binary trees can be constructed on a four points subset in $\mathbb{R}^3$.

References

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[2] N. Innami, S. Naya, "A comparison theorems for Steiner minimum trees in surfaces with curvature bounded below," Tohoku Math. Journal, (2012), to appear.

[3] A. O. Ivanov, A. A. Tuzhilin, *Extreme Networks Theory*, Moscow, Izhevsk: Inst. of Komp. Issl., (2003) [in Russian].