Shortest networks in Banach spaces: does existence of a shortest network for every three-point subset of X imply existence of a shortest network for every finite subset of X?

Problem stated by Natalia Strelkova

Networks in Banach spaces. Here a *network* in a Banach space is just a finite set of segments. The natural graph structure on a network is defined in the following way. The *edges* are the segments, the *vertices* are the endpoints of the segments. If two segments share an endpoint then the two corresponding edges share the corresponding vertex. A network is called *connected* if the corresponding graph is connected. The *length* $\ell(N)$ *of a network* N is the sum of the lengths of all its edges (segments), where the length of a segment $[a, b]$ is by definition $\|a - b\|$.

Now let X be a Banach space and fix a finite set $A = \{a_1, \dots, a_n\} \subset X$. We say that a network N *connects* A if N is connected and all points from A are vertices of N. Note that N may also have vertices that are not in A; we call these vertices *additional*.

Shortest networks. Consider the infimum of length functional $\mathrm{smt}(A) = \inf_N \ell(N)$ over the set of networks that connect A. The general question is whether this infimum is achieved, i.e. whether there exists a *shortest network* — a network that connects A and is shorter than any other network that connects A. It is not difficult to prove that if X is reflexive then for any $A \subset X$ there exists a shortest network. The non-reflexive case is more complicated, and the shortest network may not exist (see below).

Shortest networks with one additional point, or $r_1(A)$. Consider $r_1(A) = \inf_{x\in X} \sum_{i=1}^n \|x - a_i\|$. In fact this is minimizing network length not over the set of all networks connecting A but over a specific subset — the networks with one additional point x and the set of edges $\{[a_i, x]\}_{i=1}^n$. Is the infimum $r_1(A)$ achieved? If X is reflexive then the answer is yes. But in general this is not true, see [1], [2].

Connection between the two existence problems. Note that if A consists of three points then the problem of finding the shortest network is exactly the problem of finding an additional point x at which $r_1(A)$ is achieved.

In [3] it was shown that there exists a Banach space X and for any $n \geq 3$ a set A_n of n points in X such that there is no shortest network connecting A. The idea of the proof is to take the three-point set A_3 from [2] such that $r_1(A_3)$ is not achieved and prove that for every set A_n that is sufficiently close to A_3 in Hausdorff metric there is no shortest network connecting A_n. (Recall that ε-close in Hausdorff metric means that for any $x \in A_3$ there exists a point $y \in A_n$ such that $\|x - y\| < \varepsilon$ and for any $y \in A_n$ there exists a point $x \in A_n$ such that $\|x - y\| < \varepsilon$.)

PROBLEM. Let X be a Banach space. Suppose that for any three-point set $A_3 \subset X$ there exists a shortest network (i.e. $r_1(A_3)$ is achieved). Is it true that for any $A \subset X$ there exists a shortest network?

References

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[4] V. Kadets, "Under a suitable renorming every nonreflexive Banach space has a finite subset without a Steiner point," Matematychni Studii, 36 : 2, 197 – 200 (2011).