Random Triangles in the Plane

Problem stated in 2012 by Roman Karasev

Consider a triple of independent random points x_0, x_1, x_2 in the plane. By a random point we mean an absolutely continuous probability measure on the plane, by which this random point is distributed; though some authors considered discrete uniform distributions, when each point x_i is selected uniformly of a finite point set X_i. In fact, these two versions are limit cases of each other and we prefer the absolute continuous distributions for this text.

For a triple of random points we can consider the corresponding convex hull $\operatorname{conv}\{x_0, x_1, x_2\}$, this is a possibly degenerate *random triangle*, and ask questions about probability of covering a fixed point p by this random triangle.

In [5] (using the technique of M. Gromov [4]) it was proved that for a given triple of random points in the plane one can always find a point p such that the probability of the event $p \in \operatorname{conv}\{x_0, x_1, x_2\}$ is at least $1/6$. This means that at least one point (depending on the distributions) in the plane is covered with a positive probability (not depending on the distributions).

This result also has higher-dimensional generalizations, see [5, 6], for example. But here we ask the following question in the plane:

PROBLEM. Find the optimal constant in this theorem. Is it $1/6$ or a larger number?

In [2] it was shown that in the particular case when the random points have the same distribution, the optimal constant is $2/9$. In [5] the lower bound $2/9$ was also established for the case when two of the three distributions are the same. Intuitively it seems that for three different distributions the constant should not be less than $2/9$, but this is a guess with no rigorous proof.

Other sources of information concerning this problem are presented in the list of references.

References

[1] I. BÁRÁNY. A generalization of Carathéodory's theorem. *Discrete Math.* **40**:2–3 (1982), 141–152.

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[4] M. GROMOV. Singularities, expanders, and topology of maps. Part 2: from combinatorics to topology via algebraic isoperimetry. *Geometric and Functional Analysis* **20**:2 (2010), 416–526.

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