Is Steiner Minimal Tree Unique for a Boundary Set in General Position?

Problem stated by Alexander Ivanov and Alexei Tuzhilin

Steiner Minimal Trees. Let (X,ρ) be a metric space, V be a finite subset of X, and $G=(V,E)$ be a graph with the vertex set V and edge set E. We say that G is a *graph in the metric space* (X,ρ). If $e=\{x,y\}\in E$, then the value $\rho(e)=\rho(x,y)$ is said to be the *length of the edge* e. The sum of the lengthes of all the edges $e\in E$ is the *length of the graph* G. Consider a finite subset $M\subset X$. If $G=(V,E)$ is a connected graph in (X,ρ) such that $M\subset V$, then we say that G *joins* or *connects* M. Put $\mathrm{smt}(M)=\inf\big\{\rho(G)\mid G \text{ is a tree connecting } M\big\}$ and the value obtained is called by the *length of Steiner Mininmal Tree for* M. If $\rho(G)=\mathrm{smt}(M)$, where G is a tree connecting M, then G is referred as a *Steiner minimal tree* (or a *shortest tree*) with the *boundary* M.

Classical *Steiner Problem* consists in finding Steiner minimal trees for finite subsets of Euclidean plane and has a long history dates back to Fermat, see [1], [2]. Local structure of shortest networks is described in many situations: on Riemannian manifolds [3], in normalized spaces [4], [5], [6], in Alexandrov spaces [7]. The global structure is essentially less known. Generally, a boundary set permits several shortest networks (for example, the vertex set of a square in the plane), but as it is proved in [8], a finite subset of the plane "in general position" (i.e. for an open everywhere dense set of n-element subsets of the plane) permits unique shortest tree.

Problem Statement. In the case of Riemannian manifolds, the local structure of shortest trees is very similar to the case of the plane. Namely, the edges of each such tree are shortest geodesic segments meeting in common vertices by angles more than or equal to 120°. Also, all vertices of degree 1 belong to the boundary, and w.l.g. we can assume that all the vertices of degree 2 also belong to the boundary. It follows that the maximal degree of a vertex of a shortest tree is 3.

PROBLEM. Prove the uniqueness of the shortest tree for a finite boundary set "in general position" in a Riemannian manifold.

Notice that the proof from [8] is strongly based on the local structure of the shortest tree and on the geometry of the plane. Another proof suggested in [9] is essentially more topological, but it deal with a wider class of locally minimal networks, see [2] and also can not be transferred to general case. On the other hand, the statement of Problem seems very natural and expected.

Other spaces. The same question can be stated in other spaces, first of all in Alexandrov space of bounded curvature, in particular, on the surfaces of convex polyhedra. In general normed spaces the result is not valid [4], and it is reasonable to investigate what properties of the normed space imply the uniqueness in general position.

References

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