

Cutting a Convex Figure into Six Pieces

Problem stated in 2012 by Roman Karasev

There is a variety of results about “fair” partitions of convex bodies or measures in Euclidean spaces. The famous “ham sandwich” theorem [7, 6] asserts that any n absolutely continuous probability measures in \mathbb{R}^n can be simultaneously partitioned into equal halves with a single hyperplane. This classical result is known for a long time and is usually proved using the Borsuk–Ulam theorem or similar topological techniques.

Recently, Pablo Soberón [5] (see also [3] for a more general approach) has generalized this result to the case when we want to partition n measures into m equal parts with a convex partition of \mathbb{R}^n .

Another almost elementary case of this problem is partitioning a convex figure (convex compactum) in the plane into m parts of equal areas and perimeters. Nandakumar, Ramana Rao in [4] and Bárány, Blagojević, and Szűcs in [2] considered particular cases of this problem. Finally, in [3] (see also [1] for a slightly different approach) the result was established for prime powers $m = p^k$ using the technique of Victor Vasil’ev [8].

This result is relatively easy to establish for prime m , but in the general case one cannot use a decomposition of m into prime factors (as was used for the measure partition problem in [5]) because the perimeter is *not* an additive function of convex bodies. So the simplest remaining case of the area and perimeter equipartition problem is:

PROBLEM. Is it possible to partition every convex figure $C \subset \mathbb{R}^2$ into six pieces of equal areas and perimeters?

Of course, other numbers that are not prime powers are also worth attention.

References

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