On maximum volume simplices in polytopes

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Abstract

The paper is devoted to a volume interpretation of classic approaches to Radon's theorem about partitions. Here we give a seminew proof of the theorem and show how these ideas can be applied to Lassak's theorem about a maximal parallelotope in a simplex.

1 Key lemma

Lemma 1. *Let $\mathcal{B}$ be a centrally symmetric set of points in $\mathbb{R}^d$ with $2d+2$ points and center at the origin such that any (d-1)-hyperplane through 0 does not contain more than 2d-2 points of $\mathcal{B}$. Let T be a family of simplices on vertices from $\mathcal{B}$ without a pair of opposite points as vertices. Then there are only two simplices in T containing the origin. They are symmetric to each other with respect to the origin and have the maximal volume among all simplices in T.*

Proof. Denote the points of $\mathcal{B}$ by $x_i, -x_i$, $i = 1, 2, \ldots, d+1$ for each pair of opposite vertices. Consider a simplex Δ from T with the maximum Euclidean volume. Its vertices are $\varepsilon_1 x_1$, ..., $\varepsilon_{d+2} x_{d+2}$, where all ε's are ± 1. Assume 0 is outside of Δ. Then there is some facet of Δ separating Δ from 0. Without loss of generality this facet is $\varepsilon_1 x_1 \ldots \varepsilon_d x_d$. Then the distance from $\varepsilon_{d+1} x_{d+1}$ to this facet is less than the distance from $-\varepsilon_{d+1} x_{d+1}$ to this facet. Therefore the Euclidean volume of the simplex Δ' with vertices $\varepsilon_1 x_1$, ..., $-\varepsilon_{d+2} x_{d+1}$ is greater than the Euclidean volume of Δ and we have the contradiction.

Let us show that there is only one pair of such simplices. Without loss of generality we can assume that all ε's of Δ are $+1$. Note that a hyperplane passing through 0, x_3, ..., x_{d+1} separate x_1 and x_2. Consider any other simplex from T with the origin inside. The hyperplane passing through $0, \pm x_3, \ldots, \pm x_{d+1}$ is the same as for Δ and since it should separate $\pm x_1$ and $\pm x_2$ these signs for x_1 and x_2 should be the same. Analogously for all pairs of x_i. □

Remark 2. It also follows from the proof that if we cannot increase the volume of the simplex by changing one vertex to centrally symmetric then this simplex is maximal.

From this lemma it is easy to obtain the following well-known statement.

Corollary 3. *A simplex of a maximum volume inscribed into a centrally symmetric convex body contains the center of this body.*

Remark 4. For each simplex on vertices 0, $\varepsilon_1 x_1$, ..., $\varepsilon_{j-1} x_{j-1}$, $\varepsilon_{j+1} x_{j+1}$, ..., $\varepsilon_{d+2} x_{d+2}$, where all ε's are ± 1, its Euclidean volume is the same V_j for all sets of ε's, since it's just $\frac{1}{(d+1)!}$ times the same determinant. The Euclidean volume of any simplex on $\varepsilon_1 x_1$, ..., $\varepsilon_{d+2} x_{d+2}$ (i.e. any simplex of T) is $\pm V_1 \pm V_2 \pm \ldots \pm V_{d+2}$ depending on position of 0 with respect to this simplex. In case 0 is inside this simplex the volume is $V_1 + V_2 + \ldots + V_{d+2}$. And again we get that the maximum volume of simplices in T is realized if and only if 0 is inside a simplex.

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2 Radon's theorem and basic ideas

Lemma 1 can be used for the proof of the famous Radon theorem.

Theorem 5 (Radon's theorem)**.** *Any set $\mathcal{S}$ of $d+2$ points of general position in $\mathbb{R}^d$ has a unique partition into two disjoint sets whose convex hulls intersect.*

Such partitions are called Radon's partitions.

The main idea under finding Radon's partitions is in the spirit of Sarkaria's proof of Tverberg's theorem [4] which is a generalization of Radon's theorem. We consider two d-dimensional hyperplanes R_1 and R_2 in $\mathbb{R}^{d+1}$ with similar copies $\mathcal{B}_1 \subset R_1$ and $\mathcal{B}_2 \subset R_2$ of a set $\mathcal{S}$ given. For each partition of S into two disjoint subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ we consider vertices correspondent to $\mathcal{S}_1$ in $\mathcal{B}_1$ and vertices correspondent to $\mathcal{S}_2$ in $\mathcal{B}_2$. Together these $d+2$ points form vertices of a $(d+1)$-dimensional simplex. Vice versa, each simplex with vertices in $\mathcal{B}_1$ and $\mathcal{B}_2$ not containing a vertex of $\mathcal{B}_1$ and a vertex of $\mathcal{B}_2$ correspondent to the same vertex of $\mathcal{S}$ defines a partition of $\mathcal{S}$.

Now the question is to define hyperplanes and copies of $\mathcal{S}$ in them properly and to characterize Radon's partitions in terms of simplices. We consider two close constructions. The first one is a special case from Sarkaria's proof of Tverberg's theorem. Hyperplanes R_1 and R_2 and copies of $\mathcal{S}$ are symmetric with respect to 0. We will call this case centrally symmetric. The second construction is for two parallel hyperplanes R_1 and R_2 and copies of $\mathcal{S}$ congruent up to a translation. We will call this case parallel.

3 Why maximal volumes represent Radon's partitions

Lemma 6. *A partition is Radon's if and only if in the centrally symmetric case its correspondent simplex contains* 0 *inside.*

Proof. Assume the simplex Δ correspondent to a partition of $\mathcal{S}$ into $\mathcal{S}_1$ and $\mathcal{S}_2$ does not contain 0 inside. Denote the vertices of Δ in R_1 and R_2 by Δ_1 and Δ_2 respectively. Consider any hyperplane through 0 not intersecting the simplex. Then this hyperplane divides Δ_1 and Δ_2' centrally symmetric to Δ_2, so Δ_1 and Δ_2' do not intersect. Since the partition of $\mathcal{B}$ into Δ_1 and Δ_2' is congruent to the partition of $\mathcal{S}$ into $\mathcal{S}_1$ and $\mathcal{S}_2$, this partition is not Radon's.

The converse is very similar so we just describe it briefly. If the partition is not Radon's we can divide the correspondent subsets of $\mathcal{B}$ by a d-dimensional hyperplane. Connecting this hyperplane with 0 we obtain a $(d+1)$-dimensional hyperplane that does not intersect the simplex correspondent to the partition and therefore 0 is outside of it. □

Proposition 7. *Δ is a simplex of maximal Euclidean volume from T in the centrally symmetric case $\Leftrightarrow$ a partition correspondent to Δ is Radon's.*

Proof. Follows immediately from Lemmas 1 and 6. □

By proving this proposition we simultaneously proved Radon's theorem. Since by Lemma 1 there exist two maximal simplices and since they correspond to the same partition, Radon's partition exists and is unique.

Lemma 8. *Consider a simplex in $\mathbb{R}^d$ with vertices $x_1, x_2, \ldots, x_k$ with the first coordinate* 0 *and vertices $x_{k+1}, \ldots, x_{d+1}$ with the first coordinate* 1*. Then the simplex with vertices $-x_1, -x_2, \ldots, -x_k, x_{k+1}, \ldots, x_{d+1}$ have the same Euclidean volume.*

Proof. Since parallel translations of simplex vertices in one of hyperplanes do not change the volume, we may consider the case when x_1 is the origin. The rest is obvious by the volume formula. □

Proposition 9. *Δ is a simplex of maximal Euclidean volume from T in the parallel case $\Leftrightarrow$ a partition correspondent to Δ is Radon's.*

Proof. Applying the previous lemma we can see that the volumes in the parallel and centrally symmetric cases are actually the same. Therefore, the maximal volumes in two cases occur for simplices correspondent to the same partitions and, because of Proposition 7, this proposition is proved. □

Remark 10. It also follows from Remark 2 that if we cannot increase the volume of the simplex by changing one vertex to its correspondent translate then this simplex is maximal.

4 Lassak's lemmas and more

Consider a convex polytope $Q \subset \mathbb{R}^d$ and a d-dimensional vector $\vec{v}$. A simplex Δ inside Q is called *$\vec{v}$-maximal* if by moving any vertex of Δ (but only one) parallel to $\vec{v}$ inside Q the Euclidean volume of Δ cannot be increased. A simplex Δ inside Q is called *vertex maximal* if by moving any vertex of Δ (but only one) inside Q the Euclidean volume of Δ cannot be increased. The following implications are obvious: Δ is maximal (with respect to the Euclidean volume) inside $Q \Rightarrow \Delta$ is vertex maximal inside $Q \Rightarrow \Delta$ is $\vec{v}$-maximal inside Q.

For the next lemmas we consider a prism Q with bases B_1 and B_2 and a translation vector $\vec{v}$ from B_1 to B_2.

Lemma 11. *Any $\vec{v}$-maximal simplex Δ inside Q contains points y_1 and y_2 inside such that $\overrightarrow{y_1y_2} = \vec{v}$ and this pair of points is unique.*

Proof. Firstly we prove existence of such pairs. The proof is by induction on dimension. The base of induction for $d = 1$ is obvious. Now for the inductive step we have two cases.

In the first case there is a vertex of Δ that is strictly between two bases of Q. Since any move of this vertex along $\vec{v}$-direction does not make the volume of Δ larger, the facet of Δ opposite to this vertex is parallel to $\vec{v}$. Consider the section of Q by the hyperplane containing this facet. By the induction hypothesis such pair exists for this section and this case is done.

In the second case all vertices of Δ are situated on the bases of Q. If there are two vertices x_1x_2 of Δ with $\overrightarrow{x_1x_2} = \vec{v}$ that is the vector we are looking for. If there are no such vertices, for each vertex we consider its pair from the other base of Q forming a vector congruent to $\vec{v}$. Analogously to the parallel case for Radon's theorem we consider all simplices on those vertices not containing a pair of vertices forming a vector congruent to $\vec{v}$. From Proposition 9 and Remark 10 it follows that it is maximal among those simplices and represents correspondent Radon's partition. The point of intersection for this Radon's partition belongs to both subsets of the partition. Consider a point y_1 in Δ correspondent to the point of intersection in the first subset of the partition and a point y_2 in Δ correspondent to the point of intersection in the second subset of the partition. Then $\overrightarrow{y_1y_2} = \vec{v}$ or $-\vec{v}$. In both cases we are done.

Now we prove uniqueness. If there are two such pairs $\overrightarrow{y_1y_2} = \overrightarrow{y_3y_4} = \vec{v}$, then $\overrightarrow{y_1y_3} = \overrightarrow{y_2y_4}$ which is impossible due to linear independence of simplex vectors lying in B_1 and B_2. □

Following Lassak [1] by $-d\Delta$ we denote the homothetic copy of Δ with ratio $(-d)$ and homothety center at the center of mass of Δ.

Lemma 12. *Δ is vertex maximal inside $Q \Rightarrow \Delta \subset Q \subset -d\Delta \Rightarrow \Delta$ is $\vec{v}$-maximal inside Q.*

Proof. The first implication follows directly from the fact that hyperplane in a vertex of Δ parallel to its opposite face touches Q.

For the second assume the contrary, i.e. there is a vertex a of Δ such that we can move it parallel to $\vec{v}$ inside Q and increase the volume of our simplex. Consider a hyperplane H through all other vertices of Δ. It divides $-d\Delta$ into two parts one of which is a simplex Δ_1 homothetic to Δ. Moving a inside the other part cannot increase the volume so a must be moved to this simplex.

Now for each vertex of Δ consider a vector congruent to $\vec{v}$ going through this vertex and lying in Q. One of endpoints of each vector lies between the hyperplane of $-d\Delta$ containing a and H. The second endpoint of such vector for a lies in the other open halfspace of H. Since a is the farthest vertex of Δ the ends of other vectors lie in the other halfspace too. Therefore when we slightly move all vertices of Δ in the direction of $\vec{v}$ the whole simplex must remain inside Q and subsequently inside $-d\Delta$ which is of course impossible. □

Applying Lemma 11 for a parallelotope as a prism for many directions we get the following lemma.

Lemma 13. *For a d-dimensional parallelotope P with edge vectors $\vec{v}_1, \ldots, \vec{v}_d$ and a d-dimensional simplex Δ inside P, if Δ is $\vec{v}_i$-maximal inside P for each $1 \leq i \leq d$, then for each $1 \leq i \leq d$ there exists a unique pair of points y_1^i, y_2^i inside Δ such that $\overrightarrow{y_1^iy_2^i} = \vec{v}_i$.*

From three previous lemmas we immediately get two lemmas proved by Lassak (also see [2, 3]).

Lemma 14. *If $\Delta \subset Q \subset -d\Delta$ then Δ contains a unique pair of points y_1, y_2 inside such that $\overrightarrow{y_1 y_2} = \vec{v}$.*

Lemma 15. *If P is a d-dimensional parallelotope with edge vectors $\vec{v}_1, \ldots, \vec{v}_d$ and Δ is a d-dimensional simplex such that $\Delta \subset P \subset -d\Delta$ then for each $1 \le i \le d$ there exists a unique pair of points y_1^i, y_2^i inside Δ such that $\overrightarrow{y_1^i y_2^i} = \vec{v}_i$.*

Also Lemma 13 immediately implies the following fact (see Final Remarks at [1]).

Proposition 16. *For a d-dimensional parallelotope P with edge vectors $\vec{v}_1, \ldots, \vec{v}_d$ and a d-dimensional simplex Δ inside P, if Δ is vertex maximal inside P then for each $1 \le i \le d$ there exists a unique pair of points y_1^i, y_2^i inside Δ such that $\overrightarrow{y_1^i y_2^i} = \vec{v}_i$.*

Finally, using Radziszewski's operation in the similar manner as in [1] we achieve the bound on a simplex with a special maximality condition.

Proposition 17. *For a d-dimensional parallelotope P with edge vectors $\vec{v}_1, \ldots, \vec{v}_d$ and a d-dimensional simplex Δ inside P, if Δ is $\vec{v}_i$-maximal inside P for each $1 \le i \le d$, then $Vol(\Delta) \ge \frac{1}{d!} Vol(P)$.*

References

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